Old Dominion University

University Physics I

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March 2016

1 Introduction

"All science is either physics or stamp collecting."
Ernest Rutherford

1.1 The Nature of Physics

Physics is the most fundamental and all-inclusive of the sciences, and has had a profound effect on all scientific development. Scientists of all disciplines make use of ideas, laws, methods and techniques of physics. Physics is the foundation of all science, engineering and technology. Students of many fields find themselves studying physics because of the basic role it plays in all phenomena.

Richard Feynman has a beautiful description of the nature of physics in "The Feynman Lectures on Physics." The book can be found at http://www.feynmanlectures.caltech.edu/I_toc.html

The next few paragraphs in this section are based on his book.

If you are going to learn physics, you will have a lot to study: two hundred years of the most rapidly developing field of knowledge that there is. Surprisingly enough, in spite of the tremendous amount of work that has been done for all this time it is possible to condense the enormous mass of results to a large extent—that is, to find laws which summarize all our knowledge. Even so, the laws are so hard to grasp that it is unfair to you to start exploring this tremendous subject without some kind of map or outline of the relationship of one part of the subject of science to another.

You might ask why we cannot teach physics by just giving the basic laws on page one and then showing how they work in all possible circumstances. We cannot do it in this way for two reasons. First, we do not yet know all the basic laws: there is an expanding frontier of ignorance. Second, the correct statement of the laws of physics involves some very unfamiliar ideas which require advanced mathematics for their description. Therefore, one needs a considerable amount of preparatory training even to learn what the words mean. No, it is not possible to do it that way. We can only do it piece by piece.

Each piece, or part, of the whole of nature is always merely an approximation to the complete truth, or the complete truth so far as we know it. In fact, everything we know is only some kind of approximation, because we know that we do not know all the laws as yet. Therefore, things must be learned only to be unlearned again or, more likely, to be corrected.

The principle of science, the definition, almost, is the following: The test of all knowledge is experiment. Experiment is the sole judge of scientific "truth." But what is the source of knowledge? Where do the laws that are to be tested come from? Experiment, itself, helps to produce these laws, in the sense that it gives us hints. But also needed is imagination to create from these hints the great generalizations - to guess at the wonderful, simple, but very strange patterns beneath them all, and then to experiment to check again whether we have made the right guess. This imagining process is so difficult that there is a division of labor in physics: there are theoretical physicists who imagine, deduce, and guess at new laws, but do not experiment; and then there are experimental physicists who experiment, imagine, deduce, and guess.

Now, what should we teach first? Should we teach the correct but unfamiliar law with its strange and difficult conceptual ideas, for example the theory of relativity, four-dimensional space-time, and so on? Or should we first teach the simple "constant-mass" law, which is only approximate, but does not involve such difficult ideas? The first is more exciting, more wonderful, and more fun, but the second is easier to get at first, and is a first step to a real understanding of the second idea. This point arises again and again in teaching physics. At different times we shall have to resolve it in different ways, but at each stage it is worth learning what is now known, how accurate it is, how it fits into everything else, and how it may be changed when we learn more.

1.2 Physical Quantities and Units

Physics is an experimental science, based on measurements. Theory plays a major role in understanding. We measure each physical quantity in its own units, by comparison with a standard. The standard corresponds to 1.0 unit of the quantity.

Scientists measure all sorts of things in their observations and experiments. Many quantities can be determined by measuring others and then combining the measurements according to the laws of physics.

There are very many physical quantities, but practically all physical processes, characteristics and phenomena can be expressed in terms of a small number of independent, *fundamental quantities*.

There are seven fundamental (or base) quantities forming the basis of the International System of Units, commonly known as SI units, from the French *Système International d'Unités*.

Table 1.1 Fundamental quantities and their SI units

Quantity	Units	Abbreviation
length	meter	m
time	second	s
mass	kilogram	kg
temperature	kelvin	K
electric current	ampere	A
amount of substance	mole	mol
light intensity	candela	cd

Although, the choice of the units is arbitrary (they have been defined by humans rather than prescribed by nature), the SI units is the most widely used system in the word.

For the first semester of university physics we mostly need three base units: *length, time, and mass*.

1.2.1 Length

In the late 1700s the French Academy of Sciences declared the meter to be a specific fraction (1/10,000,000) of the distance from Earth's equator to the North Pole (at sea level).

In the 1870s and in light of modern precision, a series of international conferences was held to devise new metric standards. In 1889 at the first General Conference on Weights and Measures the International Prototype *Metre* was established as the distance between two lines on a standard bar composed of an alloy of ninety percent platinum and ten percent iridium, measured at the melting point of ice. That bar was a standard from 1889 to 1960.

Today the meter is defined by the distance light travels in a vacuum in 1/299,792,458 of a second. Thus, the meter is based on postulated speed of light.

Historical context of the meter can be found at http://physics.nist.gov/cuu/Units/meter.html

1.2.2 Time

Between middle ages and 1960 the second was defined as 1/86,400 of a mean solar day. The exact definition of "mean solar day" was left to astronomical theories. However, measurement showed that irregularities in the rotation of the Earth could not be taken into account by the theory and has the effect that this definition does not allow the required accuracy to be achieved.

Now the second is defined as the time it takes for 9,192,631,770 periods of the transition between two split levels of the ground state of the cesium-133 atom.

By the way, in science we still have troubles to have a good definition of time. Webster defines "a time" as "a period," and the latter as "a time," which doesn't seem to be very useful. Here are a couple quotes from great scientists: “The only reason for time is so that everything doesn't happen at once” Albert Einstein, "Time is what happens when nothing else happens" Richard Feynman.

1.2.3 Mass

At the end of the 18th century, a kilogram was the mass of a cubic decimeter (1 liter) of water. In 1889, the 1st The General Conference on Weights and Measures (*Conférence Générale des Poidset Mesures, CGPM*) sanctioned the international prototype of the kilogram, made of platinum-iridium, and declared: This prototype shall henceforth be considered to be the unit of mass.

The 3d CGPM (1901), in a declaration intended to end the ambiguity in popular usage concerning the word "weight," confirmed that: The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.

1.2.4 How large?

Many physicists find it helpful to have an intuitive feel for the sizes of magnitudes. This is especially true if you grew up using the English system of units.

Table 1.2 For orientation

Quantity	**Units**	**Good to know**
mass	kilogram	The mass of a 1-L bottle of water
distance	meter	An average height of a man in the US is 1.8 m
distance	kilometer	If you are an average person you can walk 1 km in about 12 minutes
speed	meter/second	An average person walk with a speed of 1.4 m/s
energy	joule	An apple that falls from a table has about 1 J of kinetic energy
power	watt	An average laptop uses from 50 to 70 W

1.2.5 Derived units

Most other units are derived or based on fundamental (base) units. Examples of derived units: area (m^2), speed (m/s), and mass density (kg/m^3).

We use variables to represent the values of physical quantities and relationships between them. For many quantities we use standard notations (letter, symbols), like m for mass, v for velocity, t for time, p for momentum, E for energy, ω for angular speed, etc.

1.2.6 The British system of units

These units (also called British Imperial system of units) are used only in the United States and remain in limited use in India, Malaysia, Sri Lanka, Hong Kong, and some Caribbean islands. British units are now officially defined in terms of SI units as follow *Length*: 1 inch = 2.54 cm (exactly), *Force*: 1 pound = 4.448221615260 newtons (exactly). The British unit of time is the second, defined the same way as in SI. There is no British system of electrical units. The British system has very complicated relations between base and derived units.

Table 1.3 Linear measures in the British system of units

Unit 1	**Unit 2**
12 inches (in)	1 foot (ft)
3 feet	1 yard (yd)
5 1/2 yards	1 rod (rd)
40 rods	1 furlong (fur) = 220 yards = 660 ft
8 furlongs	1 statute mile (mi) = 1,760 yards
5,280 feet	1 statute or land mile

From lectures of Professor Lewin (Massachusetts Institute of Technology) "I find it extremely difficult to work with inches and feet. It's an extremely uncivilized system. I don't mean to insult you, but think about it - 12 inches in a foot, three feet in a yard. Could drive you nuts".

Going a bit beyond nuisance of the British system of units - think about it. What is the first day of a week? If it is Sunday why do we call it weekend!

Note: you should try to think in SI units as much as you can!

1.3 Unit Prefixes

In physics, we explore the very small to the very large. The very small is a small fraction of a proton and the very large is the universe itself. For example, the horizontal size of the Universe is about 2.6*10^{26} m, the size of an electron is about 5.6*10^{-15} m. They span 45 orders of magnitude. In scientific notations: 1000,000,000,000,000,000,000,000,000,000,000,000,000,000,000 = 1.0*10^{45}. Once we have defined the fundamental units, it is easy to introduce large and smaller units for the same physical quantities. In the metric system these other units are related to the fundamental units by multipliers of 10 or 1/10.

Table 1.4 Prefixes

Factor	**Prefix**	**Symbol**	**Factor**	**Prefix**	**Symbol**	**Factor**	**Prefix**	**Symbol**
$\mathbf{10^{-24}}$	yocto	y	10^{-3}	milli	m	10^{9}	giga	G
$\mathbf{10^{-21}}$	zepto	z	10^{-2}	centi	c	10^{12}	tera	T
$\mathbf{10^{-18}}$	attpo	a	10^{-1}	deci	d	10^{15}	peta	P
$\mathbf{10^{-15}}$	femto	a	10^{1}	deka	da	10^{18}	exa	E
$\mathbf{10^{-12}}$	pico	p	10^{2}	hecto	h	10^{21}	zetta	Z
$\mathbf{10^{-9}}$	nano	n	10^{3}	kilo	k	10^{24}	yotta	Y
$\mathbf{10^{-6}}$	micro	μ	10^{6}	mega	M	10^{100}	googol	

Thus, 1 kilometer (1 km) is 1000 meters (1 km = 10^3 m), 1 centimeter (1 cm) is 1/100 meter (1 cm = 10^{-2} m).

The names of additional units are derived by adding a prefix to the name of the fundamental unit.

Attention: Don't drop the prefixes. For example 700 nm is less than 0.7 m.

Prefixes are a convenient way to express large and small numbers, but use them with care. You are guaranteed consistency when all of the numbers you are entered into a calculation are in the SI units. For example, in calculations use meters not kilometers.

1.4 Unit Consistency and Conversions

We use equations to express relationships among physical quantities, represented by algebraic symbols. Each symbol always represents both a number and a unit.

1.4.1 Dimensional Analysis

An equation must be dimensionally consistent. Dimensional analysis is a powerful technique that can help you quickly determine how likely it is that you have done a problem correctly. You check that the dimensions of you algebraic answer matches what you expect before you substitute values to computer a numerical result.

Any mechanical quantity can be represented as $[A] = M^x L^y T^z$.

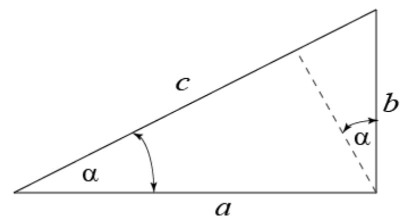
Table 1.5 Dimensions of Some Mechanical Quantities

Quantity	**Dimension**	**Units**
length	L	m
time	T	s
mass	M	kg
velocity	$L \cdot T^{-1}$	$m \cdot s^{-1}$
acceleration	$L \cdot T^{-2}$	$m \cdot s^{-2}$
volume	L^3	m^3
density	$M \cdot L^{-3}$	$kg \cdot m^{-3}$
force	$M \cdot L \cdot T^{-2}$	$kg \cdot m \cdot s^{-2}$ =newton
energy	$M \cdot L^2 \cdot T^{-2}$	$kg \cdot m^2 \cdot s^{-2}$ =joule

Example 1: The period of a simple pendulum, the time for one complete oscillation, is given by $T = 2\pi\sqrt{L/g}$, where L is the length of the pendulum and g is the acceleration due to gravity. Show that the dimension is consistent.

$$T = \sqrt{\frac{L}{(L \cdot T^{-2})}} = \sqrt{T^2} = T$$

Example 2: A proof of Pythagorean Theorem using dimensional analysis. The area A of the right-angle triangle is a function of the angle and the hypotenuse (for a right-angled triangle, only the hypotenuse length and one of the angles are needed to completely specify the triangle), or $A_c = f(c, \alpha)$. Since area's dimension is $[Area] = L^2$, then $f(c,\alpha) = c^2 g(\alpha)$, where $g(\alpha)$ is a dimensionless function of the angle.



For smaller triangles inside the original one we can write $A_a = a^2 g(\alpha)$ and $A_b = b^2 g(\alpha)$. It is obvious that $A_c = A_a + A_b$ or $c^2 g(\alpha) = a^2 g(\alpha) + b^2 g(\alpha)$, then $c^2 = a^2 + b^2$.

1.4.2 Unit Conversion

We often need to change the units in which a physical quantity is expressed. We do so by a method called *chain-link conversion*. In this method we multiple the original value by a conversion factor (a ratio of units that is equal to unity). For example, 1 min = 60 s, then (1 min/60 s) =1 as well as (60 s/1 min) =1.

Example: Let's find number of minutes in 150 seconds:

correct: $150\ s = 150\ s \cdot 1 = 150\ s\left(\frac{1\ min}{60\ s}\right) = 2.5\ min$

incorrect: $150\ s = 150\ s \cdot 1 = 150\ s\left(\frac{60\ s}{1\ min}\right) = 9000\ min$ because you get $9000\ s^2/min$

Attention 1: to ensure that you have written the conversion factor properly, check that the units cancel as necessary between numerator and denominator.

Attention 2: Some conversion cannot be easily carried out in a single step. Then, write each phase of a conversion separately.

Example 1: There is no speed limit on the German autobahn, but recommended top speed is 130 km/h. Let's express this speed in miles per hour and meters per second, where 1 mile = 1.609 km = 1609 m, 1 km = 1000 m, 1 h = 3600 s.

$$130\ km/h = \left(\frac{130\ km}{1\ h}\right)\left(\frac{1\ mile}{1.609\ km}\right) = 80.8\ mph$$

$$130\ \ km/h = \left(\frac{130\ km}{1\ h}\right)\left(\frac{1000\ m}{1\ km}\right)\left(\frac{1\ h}{3600\ s}\right) = 36.1\ \ m/s$$

Example 2: How many square centimeters in a square meter? (Note that 1 m = 100 cm)

$$1\ m^2 = (1\ m)^2 = \left[1\ m\left(\frac{100\ cm}{1\ m}\right)\right]^2 = [100\ cm]^2 = 10{,}000\ cm^2$$

1.5 Uncertainty and Significant Figures

Measurements always have uncertainties. The uncertainty is also called the error because it indicates the maximum difference there is likely to be between the measured value and the true value. We often indicate the accuracy of a measured value with the symbol ±, i.e. is a length of a pencil is given as 56.47 ± 0.02 mm, this means that the true value is unlikely to be less than 56.45 mm or greater than 56.49 mm.

There are statistical methods for determining the error in a calculation that are beyond the scope of this course. We will use a simplified approach called "significant figures."

For example, a distance is given as 137 km. It has three significant figures. By this we mean that the first two digits are known to be correct, while the third digit is uncertain, and the uncertainty is about 1 km.

Example: How many miles in 2000 meters? (1 mile = 1609 m)? In this example, when we say that the distance is 2000 meters, we mean, first, that it is neither 1999 meters nor 2001 meters, and, second, that we do not bother if the distance is more precisely, say, 1999 meters and 70 centimeters: we round it up to 2000 meters. In other words, 2000 meters in this context means some number between 1999.5 and 2000.5

Calculations give

$\frac{2000.5}{1609} = 1.243318831572405$ and $\frac{1999.5}{1609} = 1.242697327532629$

Comparing these numbers, we conclude that we should write $2000\,m\left(\frac{1\,mile}{1609\,m}\right) = 1.243\,mile$ discarding further insignificant figures.

1.5.1 Significant figures in multiplication or division

Suppose that we measured one side of a rectangle and obtained that it equals 5.77 cm. The other side, for some reason, we measured with a cruder ruler and obtained that its length is 9.9 cm. If, to find the area, we simply multiply these numbers, the calculator gives (5.77 cm)*(9.9 cm) = 57.123 cm^2.

However, realizing that we deal with rounded numbers, to check what are possible outcomes, we multiply the lower admissible values (5.765 cm)*(9.85 cm) = 56.78525 cm^2 and also the higher admissible values (5.775 cm)*(9.95 cm) = 57.46125 cm^2. Thus, we have a rather wide range of values for the area. Keeping more than two figures clearly makes no sense. So, discarding insignificant figures and rounding to two figures, we obtain (5.77 cm)*(9.9 cm) = 57 cm^2, where "57" is the properly rounded version of the original "57.123". For further uses, "57" should be understood as a number between 56.5 and 57.5.

A simple inspection shows that uncertainty in the value of the product is determined mainly by the uncertainty in the value 9.9 cm of the least precise measurement.

Rule of thumb: do not keep more figures than in the least precise input term.

1.5.2 Significant figures in addition

In the same way, if we simply add together two sides, we obtain (5.77 cm) + (9.9 cm) = 15.67 cm.

However, since we know that the length of the second side is in fact some number between 9.85 cm and 9.95 cm, we conclude that (5.77 cm)+(9.9 cm) = 15.7 cm.

Rule of thumb: do not exceed precision of the least precise input term.

Table 1.6 Using Significant Figures

Operation	**Significant figures in result**
Multiplication or division	No more than in the number with the fewest figures example: $(0.745 \times 2.2)/3.885 = 0.42$. example: $(1.32578 \times 10^{7}) \times (4.11 \times 10^{-3}) = 5.45 \times 10^{4}$
Addition or subtraction	determined by the number with the largest uncertainty (i.e., the fewest digits to the right of the decimal point) example: 27.253 + 138.2 – 11.74 = 153.6

Leading zeros are not significant, i.e. 0.000159 carries three significant figures. *Trailing* zeros are considered significant unless the value is stated without a decimal point, i.e. the value 300 has one significant digit, the value 300. has three, the value 300.00 has five. When we calculate with very large or very small numbers, we can show significant figures much more easily by using scientific notation, sometimes called "powers-of-ten notation". For example the distance from the earth to the moon is about 384,000,000 m = 3.84×10^{8} m. In this form it is clear that we have three significant figures.

Note: In most textbooks most numerical answers are given with three significant figures.

1.6 Estimates and Order of Magnitude

Quite often we face of the following situations

- A problem in hand is too complicated to be solved accurately but we need some idea about a possible solution and we need it in reasonable time.
- It seems that we do not have all the information for answering a question
- We do not need as exact as possible solution but a guess can be useful even if it is uncertain by a factor of two or even ten.
- We have solved a problem but we want to check if the solution to the problem is reasonable.
- We are going to do something, and we need to estimate quickly either needed resources (time, money, materials) or possible outcome.
- We cannot find an answer (or a credible link) asking Google

Then we can (and should) use "order-of-magnitude estimate". In many cases, the order of magnitude of a quantity can be estimated using reasonable assumptions and simple calculations. The physicist Enrico Fermi was a master at using order-of-magnitude estimations to generate answers for questions that seemed impossible to calculate because of complexity or lack of information.

Using order-of-magnitude estimations is not restricted to science but is commonly used in engineering, business, medicine, practically anywhere. When you master the order-of-magnitude estimations then you can see that very many episodes in movies from Hollywood are far from reality.

For using the order-of-magnitude estimations we need to

1. come up with as simple as possible model for our problem or question ("Make things as simple as possible, but not simpler" - Albert Einstein). Breaking dawn a problem into easier smaller problems may work as well.
2. figure out what data do we need for our model
3. get the required data using present knowledge, common sense, educated guess (by bracketing missing data and applying either geometric mean) or asking Google if possible
4. carry out (normally very simple) calculations

If we feel that our order-of-magnitude estimation is sensible we can stop here. There are a couple principal reasons when we fail, namely, oversimplified or wrong model and wrong estimation for data.

Here are a couple of such questions that can be answered using reasonable assumptions.

- How much money one needs to drive from Norfolk, VA to Los Angeles, CA?
- How much coffee is consumed daily by ODU students?
- What is the radius the radius of Earth?

For mastering the art of estimation one may read

- "Guesstimation 2.0: Solving Today's Problems on the Back of a Napkin"
  by Lawrence Weinstein, Princeton University Press (2012)
- "How Many Licks?: Or, How to Estimate Damn Near Anything"
  by Aaron Santos, Running Press (2009)

One may find interesting to read "How to Measure Anything: Finding the Value of Intangibles in Business" by Douglas W. Hubbard, 3rd edition, Wiley (2014)

1.7 Vectors

1.7.1 Coordinate systems

Very many quantities in physics deal with locations in space, for example, a position of an object at different moments in time. We need to define a coordinate system to describe the position of a point in space relative to some origin. There are multiple types of coordinate systems. The most popular systems in physics are Cartesian, polar, cylindrical, and spherical coordinate systems.

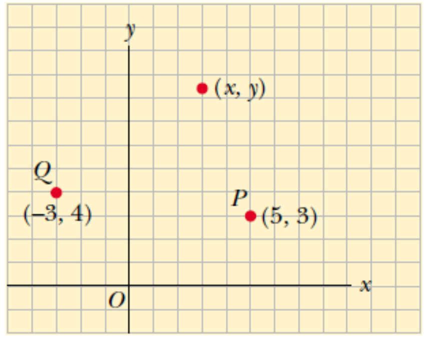


Figure 1 Designation of points in a Cartesian coordinate system. Every point is labeled with coordinates(x, y).

Cartesian coordinates are also called rectangular coordinates. The origin corresponds to a point with coordinates (0,0).

Sometimes it is more convenient to represent a point in a plane by its plane polar coordinates (r, θ), as shown in the next figure.

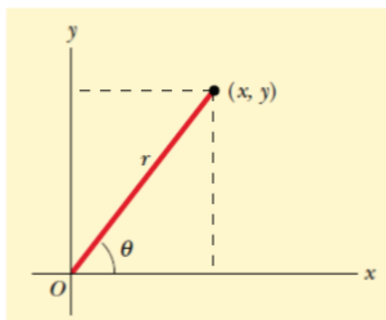


Figure 2 The plane polar coordinates of a point are represented by the distance r and the angle θ.

In this polar coordinate system, r is the distance from the origin to the point having Cartesian coordinates (x, y), and $\theta$is the angle between r and a fixed axis. This fixed axis is usually the positive x axis, and $\theta$is usually measured counterclockwise from it. From trigonometry, one can easily find that

$$x = r\cos\theta \tag{1.1}$$

$$y = r\sin\theta \tag{1.2}$$

and correspondingly

$$\mathrm{r} = \sqrt{x^2 + y^2} \tag{1.3}$$

$$\theta = \operatorname{atan}\left(\frac{y}{x}\right) \tag{1.4}$$

Attention: These four expressions relating the coordinates (x, y) to the coordinates (r, θ) apply only when θ is defined as an angle measured counterclockwise from the positive x axis.

Note that in this chapter we work with two dimensional coordinates (x, y). A generalization for three dimensional (x, y, z) system is straightforward.

1.7.2 Scalars and vectors

In our daily lives we deal, as a rule, with quantities that are completely specified by its magnitude, a single number, together with the units in which it is measured. Such a quantity is called a *scalar* and examples include temperature, time, and density.

However, there are very many physical quantities that require both a *magnitude* (≥ 0) and a *direction in space* to specify them completely. They are caller *vectors*. A familiar example is force, which has a magnitude (strength) and a direction of application. Vectors are also used to describe physical quantities such as velocity, displacement, momentum, electric field, and many more. A vector is usually indicated by either an arrow over a letter representing a physical quantity (e.g. $\vec{a}$) or by a boldface letter (e.g. $\boldsymbol{a}$). A vector can be conveniently represented as an arrow in space.

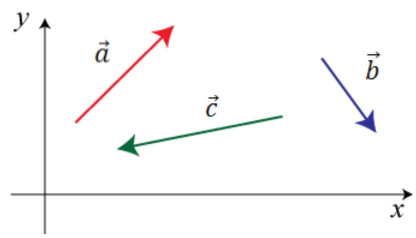


Figure 3 Three vectors in x, y plane

The length of the arrow representing a vector $\vec{a}$ is called the length or the *magnitude* of a (written as $|a|$ or just a where $a \geq 0$). Note the use of a to means the magnitude of $\vec{a}$; for this reason it is important to make it clear whether you mean a vector or its magnitude (which is a scalar). The magnitude together with the angles provides a complete description of a vector. For example, in a two-dimensional case a set of the two numbers a and θ uniquely describe vector $\vec{a}$.

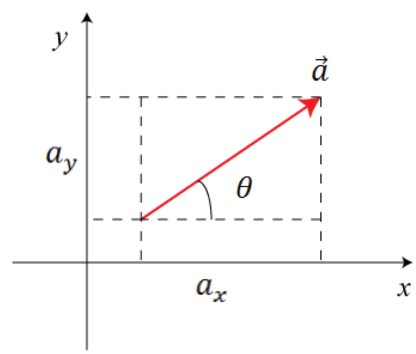


Figure 4 A vector in xy plane.

The same vector can also be uniquely described with the components of the vector a_x, a_y where

$$a_x = a\cos\theta \tag{1.5}$$

$$a_y = a\sin\theta \tag{1.6}$$

1.7.3 Addition and subtraction of vectors

So far we only need to learn how to add two (or more vectors) and to multiple a vector by a scalar. Vector products (dot and cross) will be introduced later.

Two vectors $\vec{a}$and $\vec{b}$ are defined to be equal if they have the same magnitude and point in the same direction. That is, $\vec{a} = \vec{b}$ only if $a = b$ and if $\vec{a}$ and $\vec{b}$ point in the same direction along parallel lines.

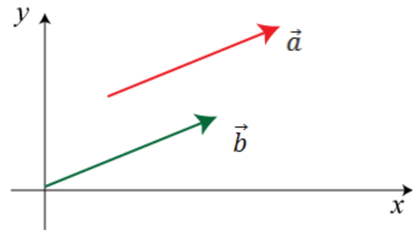


Figure 5 These two vectors are equal because they have equal lengths and point in the same direction.

For example, two vectors in Figure 5 are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

The rules for adding vectors are conveniently described by geometric methods. To add vector $\vec{b}$ to vector $\vec{a}$, first draw vector $\vec{a}$, with its magnitude represented by a convenient scale, on graph paper and then draw vector $\vec{b}$ to the same scale with its tail starting from the tip of $\vec{a}$, as shown in Figure 6. The resultant vector $\vec{c} = \vec{a} + \vec{b}$ is the vector drawn from the tail of $\vec{a}$ to the tip of $\vec{b}$.

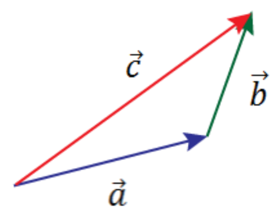


Figure 6 A vector sum $\vec{c} = \vec{a} + \vec{b}$ (the triangle method of addition).

When two vectors are added, the sum is independent of the order of the addition. (This fact may seem trivial, but as you will later, the order is important when vectors are multiplied). This can be seen from the geometric construction above and is known as the commutative law of addition:

$$\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}. \tag{1.7}$$

An alternative graphical procedure for adding two vectors is called the parallelogram rule of addition. In this construction, the tails of the two vectors $\vec{a}$ and $\vec{b}$ are joined together and the resultant vector $\vec{c}$ isthe diagonal of a parallelogram formed with $\vec{a}$ and $\vec{b}$ as two of its four sides.

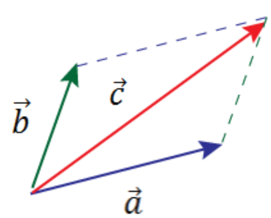


Figure 7 A vector sum $\vec{c} = \vec{a} + \vec{b}$ (the parallelogram method of addition).

The negative of the vector $\vec{a}$ is defined as the vector that when added to $\vec{a}$ gives zero for the vector sum, that is $\vec{a} + (-\vec{a}) = 0$. The vectors $\vec{a}$ and $-\vec{a}$ have the same magnitude but point in opposite directions.

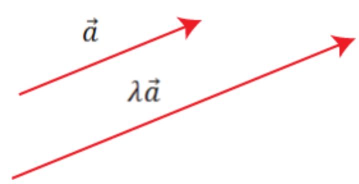
The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation $\vec{a} - \vec{b}$ as vector $-\vec{b}$ added to vector $\vec{a}$:

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right). \tag{1.8}$$

If vector $\vec{a}$ is multiplied by a positive scalar quantity n, then the product $n\vec{a}$ is a vector that has the same direction as $\vec{a}$ and magnitude na. If vector $\vec{a}$ is multiplied by a negative scalar quantity $-n$, then the product $-n\vec{a}$ is directed opposite $\vec{a}$.

1.7.4 Multiplication by a scalar

Multiplication of a vector by a scalar (not to be confused with the 'scalar product', to be discussed in section 6.3) gives a vector in the same direction as the original but of a proportional magnitude. This can be seen in figure.



The scalar may be positive, negative or zero. (It can also be complex in some applications). Clearly, when the scalar is negative we obtain a vector pointing in the opposite direction to the original vector. Having defined the operations of addition, subtraction and multiplication by a scalar, we can now introduce unit vectors and components.

1.7.5 Unit vectors and components of a vector

While geometric methods for adding or subtracting vectors are rather simple, they are not practical for solving problems. Using vector components is a much more accurate way with less room for making a mistake.

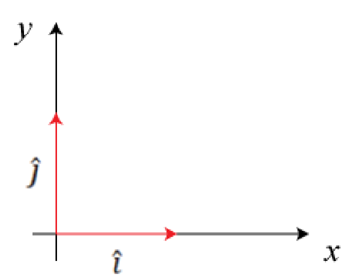
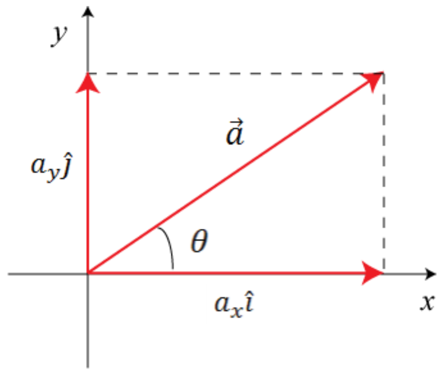


Figure 8 Two unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

A unit vector is a vector that has a magnitude of exactly 1 and points in a particularly direction. It lacks both dimensions and unit. Its sole purpose is to point – that is, to specify a direction. The unit vectors in the positive directions of the x, y and z axes are labeled as $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

Consider a vector $\vec{a}$ lying in the xy plane and making an arbitrary angle θ with the positive x axis.



This vector $\vec{a}$ may then be written as a sum of two vectors $a_x\hat{\imath}$ and $a_y\hat{\jmath}$ (remember Figure 7 - the parallelogram rule for adding two vectors), each parallel to a different coordinate axis

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} \tag{1.9}$$

A vector in two-dimensional space thus requires two components to describe fully both its direction and its magnitude. For example, a displacement in space may be thought of as the sum of displacements along the x, and y directions.

Let's remind here the definitions for the vector components (equations (1.5) and (1.6))

$$a_x = a\cos\theta, \quad a_y = a\sin\theta$$

These components can be positive or negative. Note that the signs of the components a_x and a_y depend on the angle θ. When solving problems, you can specify a vector $\vec{a}$ either with its components a_x and a_y or with its magnitude and direction a and θ.

1.7.6 Vector algebra with vector components

We can consider the addition and subtraction of vectors in terms of their components. The sum of two vectors $\vec{a}$ and $\vec{b}$ is found by simply adding their components, i.e.

$$\vec{c} = \vec{a} + \vec{b} = a_x\hat{\imath} + a_y\hat{\jmath} + b_x\hat{\imath} + b_y\hat{\jmath} = (a_x + b_x)\hat{\imath} + \left(a_y + b_y\right)\hat{\jmath} = c_x\hat{\imath} + c_y\hat{\jmath}$$

We see that the components of the resultant vector $\vec{c}$ are

$$\begin{aligned} c_x &= a_x + b_x \\ c_y &= a_y + b_y \end{aligned} \tag{1.10}$$

And their difference of two vectors can be written by subtracting their components,

$$\vec{c} = \vec{a} - \vec{b} = a_x\hat{\imath} + a_y\hat{\jmath} - b_x\hat{\imath} - b_y\hat{\jmath} = (a_x - b_x)\hat{\imath} + \left(a_y - b_y\right)\hat{\jmath} = c_x\hat{\imath} + c_y\hat{\jmath}$$

$$
\begin{aligned}
c_x &= a_x - b_x \\
c_y &= a_y - b_y
\end{aligned} \tag{1.11}
$$

We obtain the magnitude of $\vec{c}$ and the angle it makes with the x axis from its components, using the relationships

$$
c = \sqrt{c_x^2 + c_y^2} = \sqrt{(a_x + b_x)^2 + \left(a_y + b_y\right)^2} \tag{1.12}
$$

$$
\tan\theta = \frac{c_y}{c_x} = \frac{a_x + b_x}{a_y + b_y} \tag{1.13}
$$

Multiplication of a vector by a scalar λ is written as

$$
\vec{c} = \lambda\vec{a} = \lambda a_x \hat{\imath} + \lambda a_y \hat{\jmath} \tag{1.14}
$$

Note: Scalars and vectors do not change their basic properties if the coordinate system used to describe them is rotated. This is fundamentally their most important feature. The laws of physics written in terms of scalars and vectors do not change simply because we choose to change the orientation of our coordinate systems.

2 Motion in One Dimension

2.1 Motion

Many people would like to place the beginnings of physics with the work done 400 years ago by Galileo, and to call him the first physicist. Until that time, the study of motion had been a philosophical one based on arguments that could be thought up in one's head. Most of the arguments had been presented by Aristotle and other Greek philosophers, and were taken as "proven." Galileo was skeptical, and did an experiment on motion which was essentially this: He allowed a ball to roll down an inclined trough and observed the motion. He did not, however, just look; he measured *how far* the ball went in *how long a time*. By the way, Galileo's first experiments on motion were done by using his pulse to count off equal intervals of time.

In order to find the laws governing the various changes that take place in bodies as time goes on, we must be able to describe the changes and have some way to record them. The simplest change to observe in a body is the apparent change in its position with time, which we call motion. Let us consider some solid object with a permanent mark, which we shall call a point, which we can observe. We shall discuss the motion of the little marker, which might be the radiator cap of an automobile or the center of a falling ball, and shall try to describe the fact that it moves and how it moves.

These examples may sound trivial, but many subtleties enter into the description of change. Some changes are more difficult to describe than the motion of a point on a solid object, for example the speed of drift of a cloud that is drifting very slowly, but rapidly forming or evaporating.

The study of the motion of objects and the related concepts of force and energy form the field called mechanics. Mechanics is customarily divided into two parts: *kinematics*, which is the description of how objects move without regard to its cause, and *dynamics*, which deals with forces and why objects move as they do, thus dynamics studies principles that relate motion to its cause.

So far we are going to examine some general properties of a motion that is restricted in the following ways.

1. Object moves without rotating. Such motion is called *translational motion*.
2. We consider the motion itself without its cause, i.e. *kinematics* of motion.
3. The motion is along a straight-line only, which is *one-dimensional (1D) motion*. The line may be horizontal, vertical, or slanted but it must be straight.
4. The moving object is either a particle (a point-like object that does not have spatial extent) or an object such that every portion moves in the same direction and at the same rate. We simply think of some kind of small objects – small, that is, compared with the distance moved.

Note that studying first motion in 1D provides a solid foundation for understanding of motion because all basic variables of motion (position, displacement, velocity, acceleration) can be easier defined and understood in 1D space.

2.2 Reference Frames, Position and Displacement

First, we need to define a frame of reference (or a coordinate system) to describe the position of a point in space. A coordinate system consists of

- An origin at a particular point in space
- A set of coordinate axes with scales and labels
- Choice of positive direction for each axis (*unit vectors*)

There are multiple types of coordinate systems: Cartesian, polar, cylindrical, spherical and more. Coordinate transformations provide formulae for the coordinates in one system in terms of the coordinates in another system. Cartesian one dimensional (1D) or two dimensional (2D) coordinate systems are typically used in general physics courses.

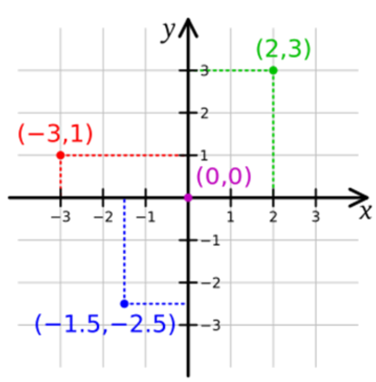
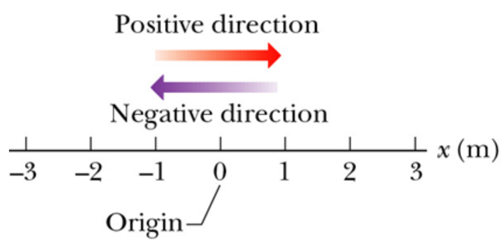


Figure 9 An example of two dimensional Cartesian coordinate systems

To locate an object means to find its position relative to some reference point, often the origin. It is clear that position of an object is a vector, since we need more than one number to locate it. Most common notation for a position vector is $\vec{r}$ that can be represented in the unit vector notations with components as $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Attention: In this chapter we will be working with motion in one dimension, when $y = 0$ and $z = 0$. Then, instead of writing $\vec{r} = x\hat{\imath}$ we will work just with the component x. This component contains both the magnitude $|x|$ and direction (positive or negative). The positive direction of the axis is the direction of increasing numbers (coordinates), which is toward the right for x axis (it corresponds to $\theta = 0^0$). The opposite direction is the negative direction (corresponding to $\theta = 180^0$).



2.2.1 *Position*

Even in one dimensional (1D) case, as we noted above, the position is a vector. Generally, we will denote the position of the object as a vector $\vec{x}$. However, we will only do it when we need to stress the vector nature of position. Most often we will denote the *position* coordinate of the object with respect to the choice of origin by $x(t)$. The position coordinate is a function of time and can be positive, zero, or negative, depending on the location of the object. Thus $+x$ means positive direction, and $-x$ is the negative one.

Figure 10 In this example the position of the red point is +2.5 m relative to the origin.

2.2.2 *Time Interval*

A *time interval* is the difference between two moments in time $\Delta t = t_2 - t_1$.

2.2.3 *Displacement*

A change from one position x_1 to another position x_2 is called a *displacement.* Displacement is a vector quantity that has both a direction and a magnitude

$$\Delta\vec{x} = \vec{x}_2 - \vec{x}_1 \tag{2.1}$$

However, as we mentioned above, in 1D case we can drop the vector symbol above vector quantities using + and – signs to identify the direction, namely $\Delta x = x_2 - x_1$.

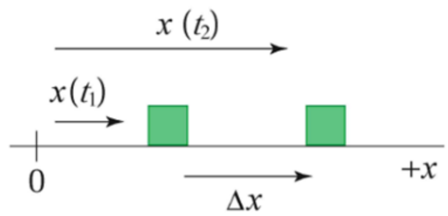


Figure 11 Positions of an object at two times t_1 and t_2 and its displacement

Note the importance of the sign, for example for $x_1 = 5$ and $x_2 = +7$ the displacement is $+7 - 5 = 2$, but for $x_1 = 5$ and $x_2 = -7$ the displacement is $-7 - (5) = -12$.

Attention: in physics "displacement" and "distance" have different definitions. Thus, "distance" is a scalar and means the total ground covered while traveling, e.g. odometer reading, but the "displacement" is a vector from where you started to where you end up.

Results of observations of motions can be conveniently presented as a table, or by means of a graph.

Examples (tables and graphs)

t(min)	s(ft)
0	0
1	1200
2	4000
3	9000
4	9500
5	9600
6	13000
7	18000
8	23500
9	24000
10	25000

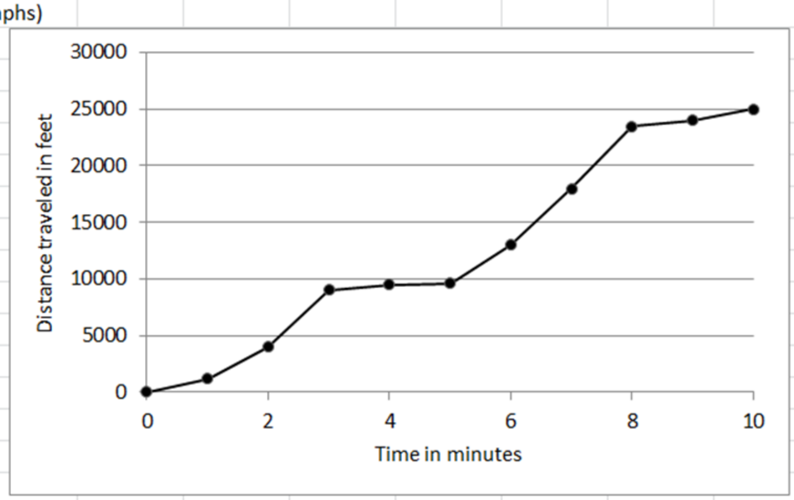


Figure 12 Example of 1D motion (position as a function of time)

2.3 Velocity and Speed

The terms *velocity* and *speed* are often used interchangeably in ordinary language. But introducing a mathematical description of motion we make a clear distinction between the two.

The term "speed" refers to how far an object travels in a given time interval regardless of direction. If a car travels 240 kilometers (km) in 3 hours, we say its average speed was 80 km/h. In general; the **average speed** of an object is defined as

$$s_{avg} = \frac{total\ distance}{t_2 - t_1} \tag{2.2}$$

Because average speed does not include direction, it lacks any algebraic sign, i.e. it is always positive.

The **average velocity** is a vector defined as "how fast", or the *displacement* divided by the time interval

$$\vec{v}_{avg} = \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{x}_2 - \vec{x}_1}{t_2 - t_1}. \tag{2.3}$$

Again, as we mentioned above, in 1D case we can drop the vector symbol above vector quantities using + and – signs to identify the direction, thus in this chapter we can use

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.4}$$

as the definition for average velocity.

The average velocity can be even equal to zero if an object ended up in the same position where it started. For example, driving from home to a class and later coming back home will result in zero displacement thus giving zero average velocity.

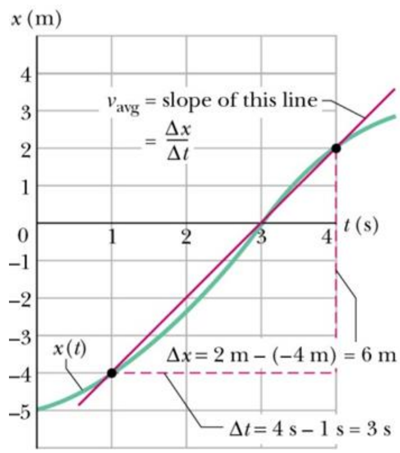


Figure 13 Calculation of average velocity

Note: Sometimes s_{avg} is the same (except for the absence of sign) as v_{avg}. However, when an object doubles back on its path the two can be quite different.

Table
Displacement versus Time

Time (s)	Displacement (m)
1	1
2	4
3	9
4	16
5	25
6	36
7	49

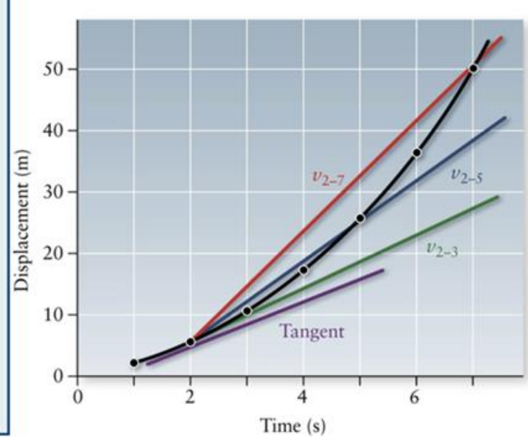


Figure 14 Average velocity at different time intervals.

In example above (Figure 14) the red, blue, and green straight lines represent the object motion as if it was moving at constant average velocity (equation (2.4)) for different time intervals. So, for various time intervals we get $v_{27} = 9\,m/s$, $v_{25} = 7\,m/s$, $v_{23} = 5\,m/s$. As the time interval becomes smaller, the lines that represent those average velocities approach the tangent to the curve at the time of interest $t = 2\,s$ and $v_{22} = 4\,m/s$.

The definitions of average speed or average velocity look as simple ones, but there are indeed some subtleties in reasoning about speed.

Example: At the point where an old lady in the car is caught by a cop, the cop comes up to her and says, "Lady, you were going 60 miles an hour!" She says, "That's impossible, sir, I was travelling for only seven minutes. It is ridiculous - how can I go 60 miles an hour when I wasn't going an hour?" How would you answer her if you were the cop?

The **instantaneous velocity** is a vector defined as "how fast" a particle is moving at a *given instant*.

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = \frac{d\vec{x}}{dt} \tag{2.5}$$

Yet again, in 1D case we can drop vector notations using + and – for directions, then we can write

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{2.6}$$

The x -component of instantaneous velocity at time t is given by the slope of the tangent line to the curve of position vs. time curve at time t

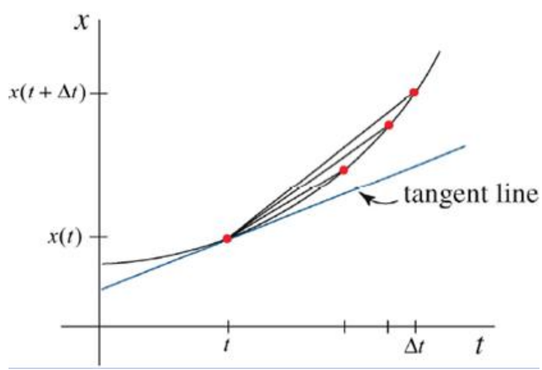


Figure 15 The instantaneous speed is the magnitude of instantaneous velocity.

2.4 Acceleration

Acceleration is the quantity that indicates how a particle's velocity changes with time (acceleration is the rate of change of velocity).

The **average acceleration** is the vector quantity that measures a change in velocity over a particular time interval.

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \tag{2.7}$$

The **instantaneous acceleration** (or simply **acceleration**) is the derivative of the velocity with respect to time

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{x}}{dt}\right) = \frac{d^2\vec{x}}{dt^2} \tag{2.8}$$

or we can write it as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} \tag{2.9}$$

Note that here we could write the second set of equations for 1D case, now without vectors, like we did before.

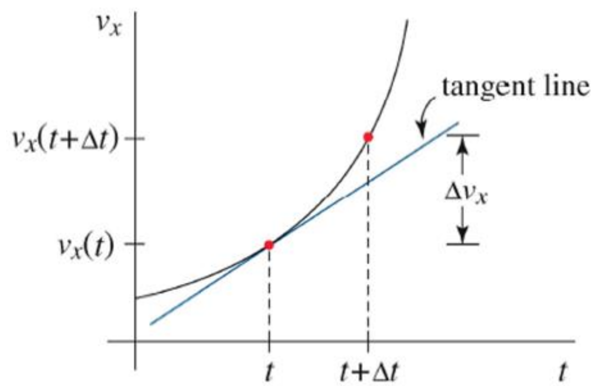


Figure 16 Instantaneous acceleration

A common unit of acceleration is meter per second per second: $m/(s{\cdot}s)$ or m/s^2. Large accelerations are sometimes expressed in terms of g units, with $= 9.8\ m/s^2$. Soon we will see that g is the free-fall acceleration.

Attention: Acceleration and velocity may have the same or different signs! If the signs are the same then an object is speeding up; if the signs are different, then an object is slowing down.

Example: The positions of two cars at successive 1.0-second time intervals are represented in the figures below.

What can conclude about the car’s speed and acceleration for the first car?

What can conclude about the car’s speed and acceleration for the second car?

2.5 Motion with constant velocity

Let's consider a simple type of motion when the velocity is constant (e.g. driving a car with 55 mph in the same direction). The acceleration is equal to zero in this case, i.e. $a = 0$. When the velocity is constant, the average and instantaneous velocity are equal, and we can write with some change in notations as

$$v = v_{avg} = \frac{x - x_0}{t - 0} \tag{2.10}$$

Here x_0 is the position at time $t = 0$, and x is the position at any later time t. We can recast this equation as

$$x = x_0 + v_{avg}t \tag{2.11}$$

As one can see, the position is a linear function of the time

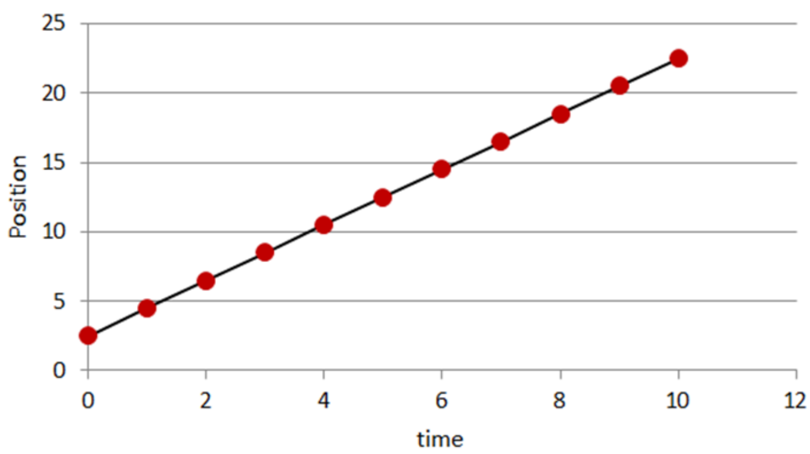


Figure 17 Position as a function of time for notion with constant velocity ($a = 0$).

2.6 Motion with constant acceleration

Many practical situations occur in which the acceleration is constant or close enough that we can assume it is constant. For example, a car accelerating after a traffic light turns green, a taking off airplane, or a falling body. In this case, the velocity changes with constant rate.

Let's recall definitions for the instantaneous velocity and acceleration

$$v = \frac{dx}{dt} \tag{2.12}$$

$$a = \frac{dv}{dt} \tag{2.13}$$

The first equation (2.12) can be written as $dx = vdt$ and the second equation (2.13) as $dv = adt$. Integrating both sides of the second equation gives $\int dv = \int adt$ with $v = at + C_1$. Since at time $t = 0$ $C_1 = v_0$ then we can write

$$v = v_0 + at$$

Now we integrate the first equation $\int dx = \int v dt$ with the equation above for the velocity $\int dx = \int (v_0 + at)dt$ to get

$$x = v_0 t + \frac{at^2}{2} + C_2$$

From the initial condition $x = x_0$ at $t = 0$ follows $C_2 = x_0$, then

$$x = x_0 + v_0 t + \frac{at^2}{2}$$

Thus, everything we need to know to describe motion under constant acceleration is contained in just two simple equations (everything else you may need for solving problems can be derived from these equation using algebra!)

$$x(t) = x_0 + v_0 t + \frac{at^2}{2} \tag{2.14}$$

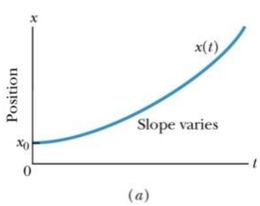
$$v(t) = v_0 + at \tag{2.15}$$

These equations are the *basic equations for motion with constant acceleration*. Reiterating again, these equations can be used to solve any constant acceleration problem in case of 1D motion.

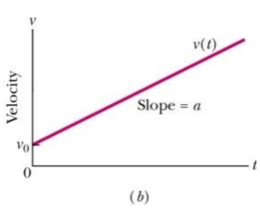
Attention: You need to have at least as many equations as unknown variables to find a unique solution. The two above equations can only be solved if there are only two unknown variables.

Just as a reminder, these two equation use $t_0 = 0$ as the reference time, so the variable t_0 does not appear in either case.

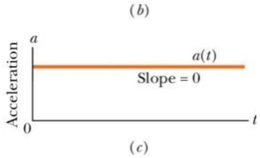
The figures below shows the position, velocity and (constant) acceleration as a function of time



The position $x(t)$ of a particle moving with constant acceleration



Its velocity $v(t)$ given at each point by the slope of the curve in (a)



Its (constant) acceleration, equal to the (constant) slope of the curve of $v(t)$

Let's consider contributions of every term in equation for position x and velocity v

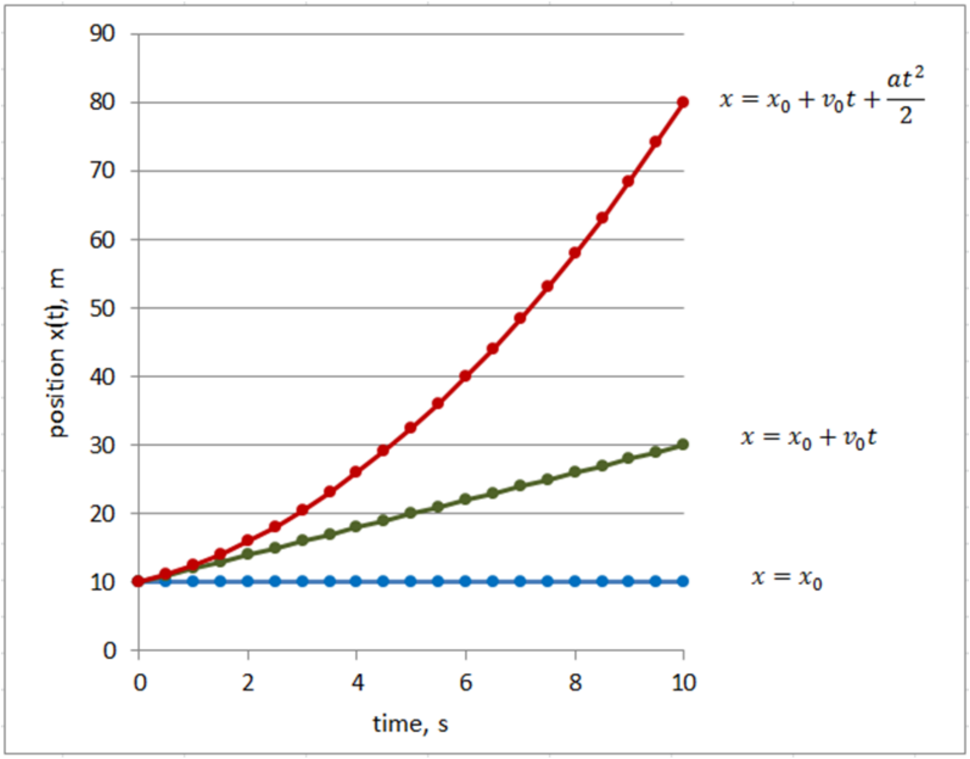


Figure 18 Contributions of terms for$x$ when $x_0 = 10\ m$, $v_0 = 2\ m\ /\ s$ and $a = 1\ m\ /\ s^2$.

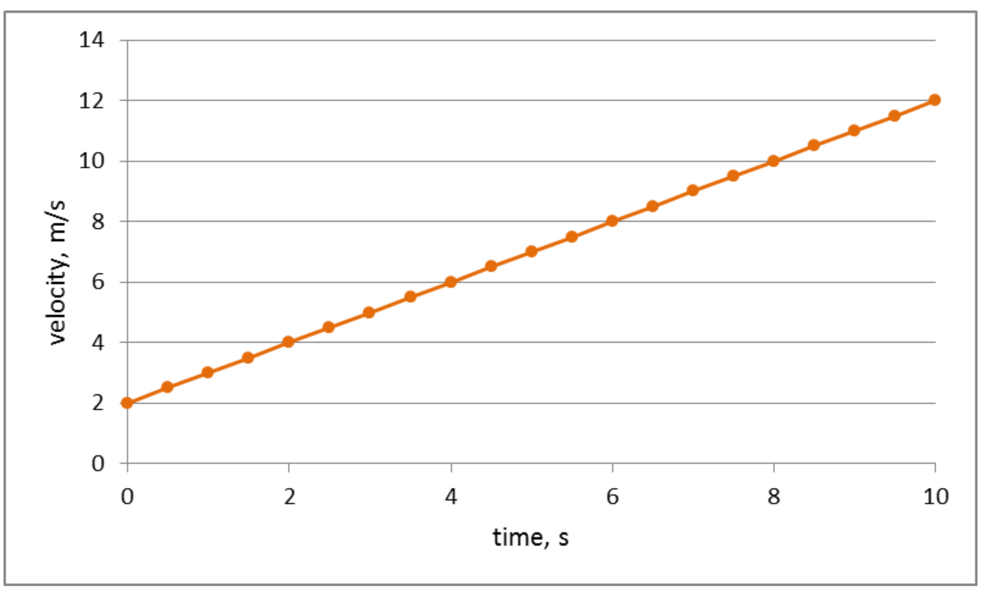
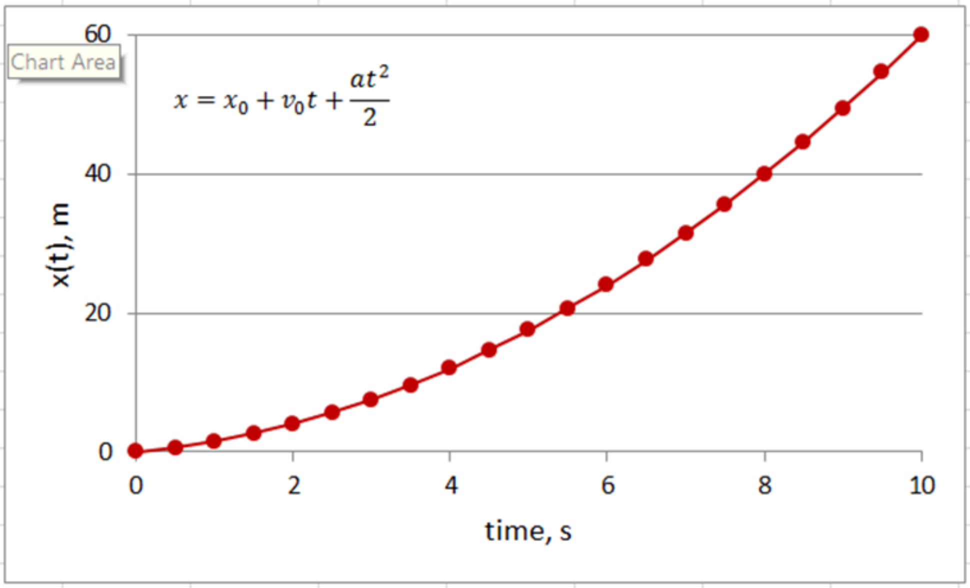


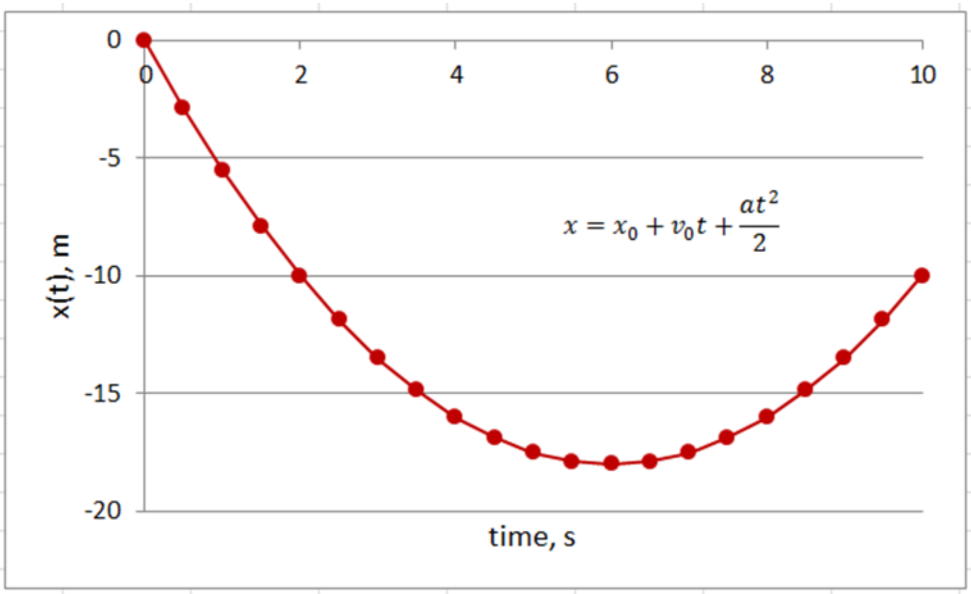
Figure 19 Velocity as a function of time $v = v_0 + at$ (for $v_0 = 2\ m\ /\ s$ and$a = 1\ m\ /\ s^2$)

Attention: Deceleration does not mean the acceleration is negative. A deceleration results in an object's speed decreasing in magnitude. An object is decelerating – slowing down – when its acceleration and velocity have opposite signs. Here are two examples.

Example 1: where $v_0 = 1\ m/s$, and $a = 1\ m/s^2$ have the same sign (direction)



Example 2: where *initially* velocity and acceleration have opposite signs $v_0 = -6\ m/s$, $a = +1\ m/s^2$ (note that after $t = 6\ s$ the velocity has the same sign as acceleration)



It is often useful to have a relationship between position, velocity and (constant) acceleration that does not involve the time. To obtain this we first solve the first basic equation for time

$$t = \frac{v - v_0}{a}$$

and then substitute the result into the second equation

$$x = x_0 + v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2$$

$$2a(x - x_0) = 2v_0v - 2{v_0}^2 + v^2 - 2vv_0 + {v_0}^2$$

and finally

$$v^2 = {v_0}^2 + 2a(x - x_0) \tag{2.16}$$

This equation is useful if we do not know t and are not required to find it (t can be called a "missing variable" in this case).

We can also eliminate the acceleration from the basic equations (2.14) and (2.15) to produce an equation in which acceleration a does not appear (a is a "missing variable")

$$x - x_0 = \frac{1}{2}(v_0 + v)t \tag{2.17}$$

The power of physics is in generalization of complicated phenomena with one or only a few equations in terms of small number of variables. Here we have our first example of that capability. Just TWO equations describe all one dimensional motion with constant accelerations.

SUMMARY: Let's write again the two basic equations describing 1D motion of a particle with constant acceleration

$$x(t) = x_0 + v_0t + \frac{at^2}{2} \tag{2.18}$$

$$v(t) = v_0 + at \tag{2.19}$$

together with the two auxiliary equations that are easily derived from the equations above, namely

$$v^2 = {v_0}^2 + 2a(x - x_0) \tag{2.20}$$

$$x - x_0 = \frac{1}{2}(v_0 + v)t \tag{2.21}$$

2.7 Freely Falling Bodies

The most familiar example of motion with (nearly) constant acceleration is a body falling under the influence of the earth's gravitational attraction. Such motion has held the attention of philosophers and scientists since ancient times. In the fourth century B.C., Aristotle thought (erroneously) that heavy bodies fall faster than light bodies, in proportion to their weight. Nineteen centuries later, Galileo argued that a body should fall with a downward acceleration that is constant and independent of its weight.

Experiment shows that if the effects of the air can be neglected, Galileo is right; all bodies at a particular location fall with the same downward acceleration, regardless of their size or weight.

If in addition the distance of the fall is small compared with the radius of the earth, and if we ignore small effects due to the earth's rotation, the acceleration is constant. The idealized motion that results under all of these assumptions is called free fall, although it includes rising as well as falling motion.

The constant acceleration of a freely falling body is called the acceleration due to gravity, and we denote its magnitude with the letter g. We will frequently use the approximate value of g at or near the earth's surface: $g = 9.8\ m/s^2$.

The exact value varies with location, so we will often give the value of g at the earth's surface to only two significant figures. Because g is the magnitude of a vector quantity, it is always a positive number.

On the surface of the moon, the acceleration due to gravity is caused by the attractive force of the moon rather than the earth, and $g = 1.6\ m/s^2$. Near the surface of the sun, $g = 270\ m/s^2$.

Attention: Objects accelerate downward under the influence of gravity, but the value of g is positive. Accordingly, the equations for the freely falling bodies are easily written using (2.18) for the position

$$y(t) = y_0 + v_0 t - \frac{gt^2}{2} \tag{2.22}$$

and (2.19) for the velocity

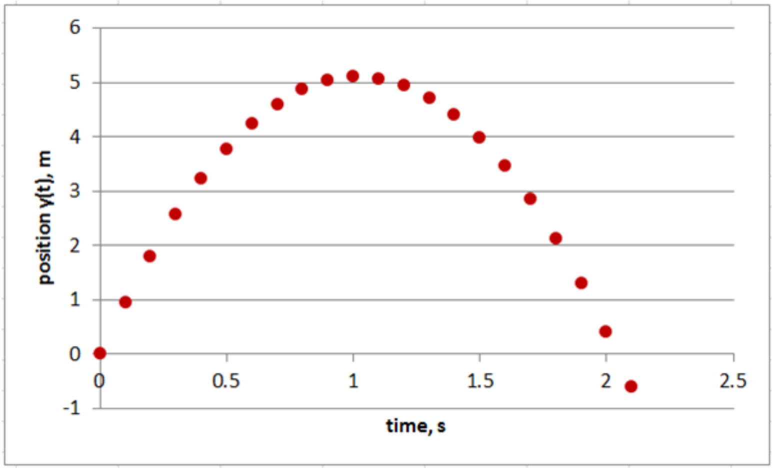
$$v(t) = v_0 - gt \tag{2.23}$$

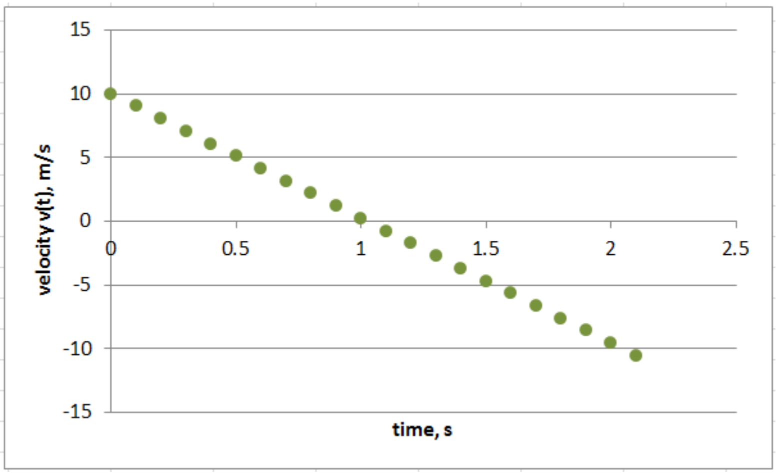
with a quite practical auxiliary equation

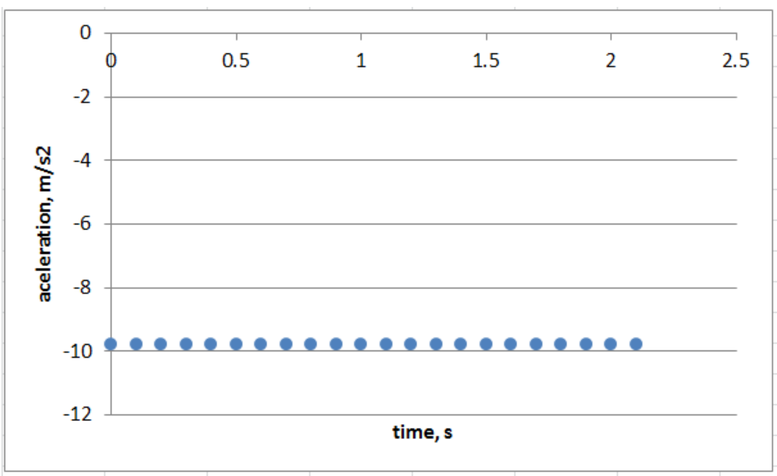
$$v^2 = {v_0}^2 - 2g(y - y_0) \tag{2.24}$$

Here is a link to a wonderful experiment -free fall for a hammer and a feather on the moon http://www.youtube.com/watch?v=5C5_dOEyAfk

Example: position, velocity and acceleration as functions of time for $v_0 = 10\ m/s\,, g = 9.8\ m/s^2$







2.8 Most common problems

Most problems in introductory physics on one dimensional motion can be classified as

Case 1: One object, one time interval
Then all we need is the two basic equations

$$x = x_0 + v_0 t + \frac{at^2}{2}$$
$$v = v_0 + at$$

Remember that the two auxiliary equations (2.20) and (2.21) are easily derived from the basic equations.

Case 2: One object, two time intervals
In this case we use the basic equations two times, first for the first time interval, and later for the second interval, where the results from the first interval are the initial conditions for the second interval. This for the first interval (from time t_0 to time t_1)

$$x_1 = x_0 + v_0 t_1 + \frac{a_0 t_1^2}{2}$$
$$v_1 = v_0 + a_0 t_1$$

and then for the second interval (from time t_1 to time t_2)

$$x_2 = x_1 + v_1 t_2 + \frac{a_1 t_2^2}{2}$$
$$v_2 = v_1 + a_1 t_2$$

Case 3: Two objects, one time interval
Then we have a system of equations for two objects that share the same time

$$x_1 = x_{01} + v_{01} t + \frac{a_1 t^2}{2}$$
$$v_1 = v_{01} + a_1 t$$
$$x_2 = x_{02} + v_{02} t + \frac{a_2 t^2}{2}$$
$$v_2 = v_{02} + a_2 t$$

There are very many variations for "two object problems". As a rule solutions can be derived from the equations above (after some simple algebra).

One of examples for such problems is a "collision" problem, when one object chases a second object, and later they are at the same point in space ($x_1 = x_2$) at the same moment in time t_c.

$$x_{01} + v_{01} t_c + \frac{a_1 t_c^2}{2} = x_{02} + v_{02} t_c + \frac{a_2 t_c^2}{2}$$

Generally, time t_c is unknown, and you need to solve quadratic equations to find it

$$\frac{a_2 - a_1}{2} t_c^2 + (v_{20} - v_{10})t_c + (x_{02} - x_{01}) = 0$$

If the initial separation between two objects is zero $x_{02} - x_{01} = 0$, then you solve a linear equation.

2.9 Examples

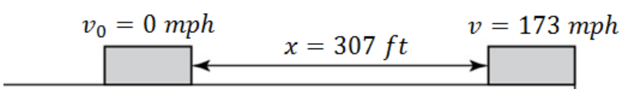
Example 2-1

The catapult of the aircraft carrier USS Abraham Lincoln accelerates an F/A-18 Hornet jet fighter from rest to a takeoff speed of 173 mph in a distance of 307 ft. Assume constant acceleration.

a) Calculate the acceleration of the fighter in m/s.
b) Calculate the time required for the fighter to accelerate to takeoff speed.

SOLUTION:

1. Physics – one-dimensional motion with constant acceleration for *one object* and *one time interval*



2. The basic equations for 1D motion with constant acceleration

$$x = x_0 + v_0 t + \frac{at^2}{2}$$

$$v = v_0 + at$$

3. Using given data $x_0 = 0\ m$ and $v_0 = 0\ m/s$, we may rewrite the basic equations as

$$x = \frac{at^2}{2}$$

$$v = at$$

4. There are two unknowns in the system above, namely the acceleration a and the time t. From the second equation we have $t = v/a$. Substituting it into the first equation gives

$$x = \frac{1}{2} \cdot a \cdot \frac{v^2}{a^2} = \frac{v^2}{2a}, \quad then \quad a = \frac{v^2}{2x},$$

using this solution with $t = v/a$

$$t = \frac{v}{a} = v \cdot \frac{2x}{v^2} = \frac{2x}{v}$$

Now we have two analytic solutions for the unknowns.

5. Calculations:

The initial data in SI units (we use 1 ft = 0.3048 m, 1 mile = 1609 m, 1 h = 3600 s)

$$307\ ft = 307\ ft\left(\frac{0.3048\ m}{1\ ft}\right) = 93.6\ m$$

$$173\ mph = 173\ \frac{mile}{h}\left(\frac{1609\ m}{1\ mile}\right)\left(\frac{1\ h}{3600\ s}\right) = 77.3\ m/s$$

calculations

$$a = \frac{v^2}{2x} = \frac{(77.3\ \ m/s)^2}{2 \times 93.6\ m} = 31.9\ \ m/s^2\ , \qquad t = \frac{2x}{v} = \frac{2 \times 93.6\ m}{77.3\ m/s} = 2.42\ s$$

6. Let's evaluate the answer.

Units and dimensions:

$$a = \frac{v^2}{2x} \rightarrow \left[\frac{m^2}{s^2} \cdot \frac{1}{m}\right] = \left[\frac{m}{s^2}\right]\ \ OK! \qquad t = \frac{2x}{v} \rightarrow \left[m \cdot \frac{s}{m}\right] = [s]\ \ OK!$$

Both the time and acceleration have proper units and dimensions.

The takeoff time $t = 2.42\ s$ looks as a reasonable numerical value.

Example 2-2

You are driving down the highway late one night at 58 mph when a deer steps into the road 50 m (about 164 ft) in front of you. Your reaction time before stepping on the brakes is 0.5 s, and the maximum deceleration of your car is 9.1 m/s^2. How much distance is between you and the deer when you come to stop?

SOLUTION:

1. Physics – one-dimensional motion with constant acceleration for *one object* but *two time intervals*

2. The basic equations for 1D motion with constant acceleration

$$x = x_0 + v_0 t + \frac{at^2}{2}$$

$$v = v_0 + at$$

3. Note that we have two phases of the motion

Phase 1: "thinking distance" or travelling with constant speed during the reaction time t_1

$$x_1 = v_0 t_1$$

Phase 2: "braking distance" or motion with constant deceleration

$$x_2 = v_0 t_2 - \frac{at_2^2}{2}$$

$$0 = v_0 - at_2$$

From the last two equations

$$x_2 = \frac{v_0^2}{2a}$$

4. The total stopping distance

$$x = x_1 + x_2 = v_0 t_1 + \frac{v_0^2}{2a}$$

5. Calculations

$$58\, mph = 55\ \frac{mile}{h}\left(\frac{1609\, m}{1\, mile}\right)\left(\frac{1\, h}{3600\, s}\right) = 25.92\, m/s$$

$$x = 25.92\, m/s \cdot 0.5\, s + \frac{(25.92\, m/s)^2}{2 \cdot 9.8\ m/s^2} = 49.9\, m$$

So the car stopped 0.1 m in front of the deer.

6. We have got both proper dimensions and reasonable numerical results.

<u>Example 2-3</u>

A car speeding at 90 mph passes a still police car which immediately takes off in hot pursuit. Assume that the speeder continues at a constant speed but the police car moves with constant acceleration. The technical specification states the police car can accelerate from 0 mph to 60 mph in 8.7 s.

a) How long would it take for the police car to overtake the speeder?
b) Estimate the distance (in meters and miles) of the hot pursuit.
c) Estimate the police car's speed at that moment the police car overtakes the speeder.

SOLUTION

1. Physics – one dimensional motion with constant acceleration for *two objects*

2. The basic equations (for two objects)

$$x_1 = x_{10} + v_{10} t + \frac{a_1 t^2}{2} \qquad x_2 = x_{20} + v_{20} t + \frac{a_2 t^2}{2}$$

$$v_1 = v_{10} + a_1 t \qquad v_2 = v_{20} + a_2 t$$

Here we call index *1* for the first object (let it be the speeder), and index *2* for the second object (the police car)

3. The basic equation can be simplified using given data and conditions, namely

The data

- at initial time $t = 0$ both cars have the same position $x_{10} = 0\, m,\ x_{20} = 0\, m$
- the speeder keeps moving with a constant speed, i.e. $a_1 = 0\, m/s^2$
- the police cars is initially at rest, i.e. $v_{20} = 0\, m/s$

The condition (the police car overtakes the speeder)

- at some time $t = t_f$ both cars are at the same position on the road, i.e. $x_1 = x_2$

Then the original basic equations can be written as

$$x_1 = v_{10}t \qquad x_2 = \frac{a_2 t^2}{2}$$

$$v_1 = v_{10} \qquad v_2 = a_2 t$$

4. Using the condition $x_1 = x_2$ we have

$$v_{10}t = \frac{a_2 t^2}{2}$$

The last equation has two unknowns, namely t and a_2. The acceleration of the police car can be found from the given data (the police car can accelerate from 0 mph to 60 mph in 8.7 s) using the definition for the average acceleration

$$a_{avg} = \frac{\Delta v}{\Delta t}$$

Then solving the equation $v_{10}t = \frac{a_2 t^2}{2}$ for the unknown time we get

$$t = \frac{2v_{10}}{a_2}$$

Having the time, we can easily find the distance of the hot pursuit

$$x_1 = v_{10} \cdot \frac{2v_{10}}{a_2} = \frac{2v_{10}^2}{a_2} \quad or \quad x_2 = \frac{a_2 t^2}{2} = \frac{a_2}{2} \cdot \frac{2^2 \cdot v_{10}^2}{a_2} = \frac{2v_{10}^2}{a_2} \quad (the\ same\ as\ x_1)$$

Thus

$$x = \frac{2v_{10}^2}{a_2}$$

Using $v_2 = a_2 t$ we get

$$v_2 = a_2 \cdot \frac{2v_{10}}{a_2} = 2v_{10}$$

$$v_2 = 2v_{10}$$

5. Calculations

First we should switch to SI units using: 1 mile = 1609 m, 1 h = 3600 s

$$90\ mph = 90\ \frac{mile}{h}\left(\frac{1609\ m}{1\ mile}\right)\left(\frac{1\ h}{3600\ s}\right) = 40\ m/s$$

$$60\ mph = 60\ \frac{mile}{h}\left(\frac{1609\ m}{1\ mile}\right)\left(\frac{1\ h}{3600\ s}\right) = 27\ m/s$$

The police car acceleration

$$a_2 = \frac{27\ m/s - 0\ m/s}{8.7\ s} = 3.1\ \ m/s^2$$

$$t = \frac{2v_{10}}{a_2} = \frac{2 \cdot 40\ m/s}{3.1\ \ m/s^2} = 26\ s$$

$$x = \frac{2v_{10}^2}{a_2} = \frac{2 \cdot (40\ m/s)^2}{3.1\ \ m/s^2} = 1030\ m$$

$$v_2 = 2v_{10} = 80\ m/s$$

6. Let's evaluate the results

The dimensions and units are correct.

Both the time and distance of the pursuit seem realistic.

How about the speed of the police car? Since we are more comfortable with mph or km/h we can write that $80\ m/s = 180\ mph$. This is very high (and risky) speed! It is rather unlikely that an average police car can go so fast (unless it is a Ferrari or Lamborghini).

Despite the numerical solutions seem correct, in real life different pursuit tactics should be used (decreasing the acceleration when the speed is above 100 mph that would result in a longer pursuit time, or calling for a roadblock ahead of the speeder, etc.).

Example 2-4

The engineer of a passenger train traveling at 30.0 m/s sights a freight train whose caboose is 100 m ahead on the same track. The freight train is traveling at 15.0 m/s in the same direction as the passenger train. The engineer of the passenger train immediately applies the brakes, causing a constant acceleration of -1.0 m/s^2, while the freight train continues with constant speed.

a) Will the cows nearby witness a collision? If so, determine how far from the initial position of the passenger train and at what time the collision occurs.
b) If not, determine the distance of closest approach between two trains

SOLUTION

1. Physics – one dimensional motion with constant acceleration for *two objects*

2. The basic equations (for two objects)

$$x_1 = x_{10} + v_{10}t + \frac{a_1 t^2}{2} \qquad\qquad x_2 = x_{20} + v_{20}t + \frac{a_2 t^2}{2}$$

$$v_1 = v_{10} + a_1 t \qquad\qquad v_2 = v_{20} + a_2 t$$

Here we call index *1* for the passenger train, and index *2* for the cargo train

3. The basic equation can be simplified using given data and conditions, namely

The data

- at initial time $t = 0$ the passenger train is at $x_{10} = 0$ but the freight train is at x_{20}
- the freight train moves with a constant speed, i.e. $a_2 = 0$

The condition (a collision)

- at some time $t = t_f$ both trains are at the same position i.e. $x_1 = x_2$

Then the original basic equations for the positions can be written as

$$x_1 = v_{10}t + \frac{a_1 t^2}{2} \qquad x_2 = x_{20} + v_{20}t$$

4. Using the condition $x_1 = x_2$ we have

$$v_{10}t + \frac{a_1 t^2}{2} = x_{20} + v_{20}t, \qquad \frac{a_1 t^2}{2} + (v_{10} - v_{20})t - x_{20} = 0$$

This is a quadratic equation for t.

5. Calculations

Solving the equation for $a = -1.0\ m/s^2$, $v_{10} - v_{20} = 15\ m/s$, and $x_{20} = 100\ m$ gives two solutions $t_1 = 10.0\ s$, $t_2 = 20.0\ s$. The first solutions corresponds to the collision. If the trains were travelling on parallel tracks, the second solution would correspond for the trains to run parallel again.

For $t_1 = 10.0\ s$ the position of the trains (relative to the point where the engineer of the passenger train saw the problem $x_{10} = 0$) is $x_1 = 250\ m$

6. Let's evaluate the results

Both the time and distance of the pursuit seem realistic.

Attention. If solving quadratic equation for time t you are getting complex roots (i.e. a negative number under square root) then there is no collision between the objects. In this case the distance between the trains as a function of time is

$$x_2 - x_1 = x_{20} + v_{20}t - v_{10}t - \frac{a_1 t^2}{2}$$

Differentiating over time t and setting $d(x_2 - x_1)/dt = 0$ gives equation to find the time for the closest approach

$$\frac{d(x_2 - x_1)}{dt} = v_{20} - v_{10} - a_1 t = 0, \quad t = \frac{v_{10} - v_{20}}{a_1}$$

As one can see, at such distance the velocities of both trains are equal, or $v_{10} + a_1 t = v_{20}$. Having this time you can easily find the distance of the closest approach between two trains.

Example 2-5

An apple (a fruit or a computer) is dropped from a bridge that is 52.0 m above the river. Neglecting air resistance

a) How long does the apple take to reach the water?
b) What is its speed just as it strikes the water surface?

SOLUTION:

1. Physics – one-dimensional motion vertical with constant free-fall acceleration ($a = -g$)

2. The basic equations

$$y = y_0 + v_0 t - \frac{gt^2}{2}$$

$$v = v_0 - gt$$

3. The basic equations can be simplified using the given conditions ($v_0 = 0, y = 0$). (We choose the river surface as our zero level). Then

$$0 = y_0 - \frac{gt^2}{2}$$

$$v = -gt$$

4. Solving the first equation gives the time

$$t = \sqrt{\frac{2y_0}{g}}$$

then from the second equation

$$v = -g \cdot \sqrt{\frac{2y_0}{g}} = -\sqrt{2y_0 g}$$

5. Calculations

$$t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2 \cdot 52\, m}{9.8\ m/s^2}} = 3.26\ s \qquad v = -\sqrt{2y_0 g} = -\sqrt{2 \cdot 52\, m \cdot 9.8\, m/s^2} = -31.9\ m/s$$

6. The time looks reasonable (from experience). We certainly have correct units for both time and velocity.

Example 2.6

If the apple was thrown vertically upward from the same bridge with a speed of 10.0 m/s

a) How high above its starting point would the apple go?

b) In how many seconds after being thrown upward would the apple strike the water below?

SOLUTION:

1. Physics – one-dimensional motion vertical with constant free-fall acceleration

2. The basic equations

$$y = y_0 + v_0 t - \frac{gt^2}{2}$$

$$v = v_0 - gt$$

3-4. The problem has two parts, namely motion to the highest point, and total motion to the river.

a) For the first part the basic equations can be rewritten as

$$y_{top} = y_0 + v_0 t - \frac{gt^2}{2}$$

$$0 = v_0 - gt$$

The first equation has two unknowns, but the second equation has only one unknown. Solving the second equation gives the time to the top point

$$t = v_0/g$$

Using this time in the first equation provides the distance

$$y_{top} = y_0 + v_0 \frac{v_0}{g} - \frac{g}{2}\frac{v_0^2}{g^2} = y_0 + \frac{1}{2}\frac{v_0^2}{g}$$

thus, from the bridge the apple goes as high as

$$\Delta y = \frac{v_0^2}{2g}$$

b) For the second part the final vertical position is zero (the river)

$$0 = y_0 + v_0 t - \frac{gt^2}{2}$$

$$v = v_0 - gt$$

The first equation has only one unknown, namely the time that we are looking for. This is a quadratic equation.

5. Calculations

$$a)\ \Delta y = \frac{v_0^2}{2g} = \frac{(10\ m/s)^\wedge 2}{2 \cdot 9.8\,m/s^2} = 5.10\,m$$

b) the quadratic equation has two solutions, $t_1 = -2.39\ s, t_2 = 4.43\ s$.

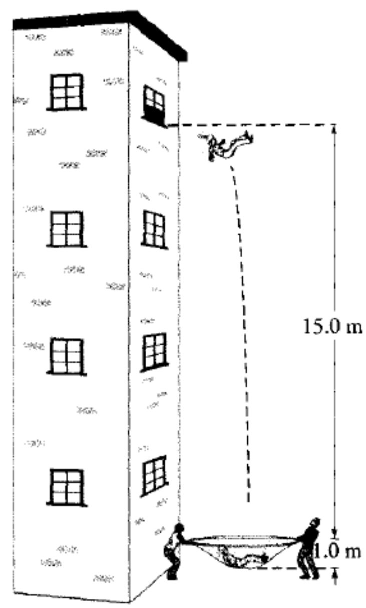
Out of two solutions only the second satisfies the given conditions that the apple hits the water after it was thrown from the bridge. The first solution satisfies another condition that the apple was at the river level before it reached the bridge with the given speed. This could happen if the apple was thrown from the water surface with an appropriate velocity.

6. The time looks reasonable (a bit large time when the apple was thrown upward). We certainly have correct units for both time and speed.

<u>Example 2-7</u>

A 75-kg person jumps from a fourth-story window 15.0 m above a firefighter's safety net. The survivor stretches the net 1.0 m before coming to rest,

a) How long was the survivor in free fall?
b) What was his speed just as he reached the net?
c) What was the average deceleration experienced by the survivor on the net (in g units)?
d) What would you do to make it "safer" (that is to generate a smaller deceleration)? Would you stiffen or loosen the net? Explain.
e) How would your answers change if it was a 1,500 kg hippopotamus?



SOLUTION

1. Physics – one dimensional motion with constant acceleration

2. The basic equations for 1D motion with constant acceleration

$$y = y_0 + v_0 t - \frac{gt^2}{2}$$

$$v = v_0 - gt$$

3. From the given information follows that we have two phases of motion, the first one is from the window to the net, and the second one is the stopping by stretching the net. Let's call the window as the initial position. The net is located at level 1, and the end position is at level 2. Then the velocity at the end of the first phase is the initial velocity for the second phase. Since the initial velocity was $0\ m/s$, and the final velocity at level 2 was $v_2 = 0\ m/s$,

then for the first phase of motion

$$y_1 = y_0 - \frac{gt_1^2}{2} v_1 = -gt_1$$

and for the second phase of motion

$$y_2 = y_1 + v_1 t_2 + \frac{a_2 t_2^2}{2} \cdot v_2 = 0 = v_1 + a_2 t_2$$

4. We can easily solve the first set of equation to find both the time t_1 and velocity v_1

$$t_1 = \sqrt{\frac{2(y_0 - y_1)}{g}} \quad and \quad v_1 = -g\sqrt{\frac{2(y_0 - y_1)}{g}} = \sqrt{2(y_0 - y_1)g}$$

Now we consider the second set of equations. From $0 = v_1 + a_2 t_2$ we have $t_2 = -v_1/a_2$ (it looks like we have a problem here with negative time, but remember that v_1 is negative!). Then the first equation reads

$$y_2 - y_1 = v_1 t_2 + \frac{a_2 t_2^2}{2} = v_1\left(-\frac{v_1}{a_2}\right) + \frac{a_2}{2}\frac{v_1^2}{a_2^2} = -\frac{v_1^2}{a_2} + \frac{v_1^2}{2a_2} = -\frac{v_1^2}{2a_2} \quad or \quad v_1^2 = 2a_2(y_1 - y_2) \; and \; a_2 = \frac{v_1^2}{2(y_1 - y_2)}$$

At this moment we can write analytic solutions for all questions

(a) How long was the survivor in free fall?

$$t_1 = \sqrt{\frac{2(y_0 - y_1)}{g}}$$

(b) What was his speed just as he reached the net?

$$v_1 = \sqrt{2(y_0 - y_1)g}$$

(c) What was the average deceleration experienced by the survivor on the net (in terms of gravity "g")?

$$a_2 = \frac{v_1^2}{2(y_1 - y_2)} = \frac{2(y_0 - y_1)g}{2(y_1 - y_2)} = \frac{(y_0 - y_1)}{(y_1 - y_2)} g$$

(d) What would you do to make it “safer” (that is to generate a smaller deceleration)? Would you stiffen or loosen the net? Explain

It is clear from the equation for a_2 that increasing the stopping distance $(y_1 - y_2)$ will decrease the deceleration, making safer landing. Therefore, loosening the net will make it "safer".

(e) How would your answers change if it was a 1,500 kg hippopotamus?

All our answer does not depend on mass of an object. Therefore, there results are going to be the same for any object if the effect of air resistance can be neglected.

5. Calculations

All the initial data were given in SI units (lucky us)

$$t_1 = \sqrt{\frac{2(y_0 - y_1)}{g}} = \sqrt{\frac{2 \cdot 15\ m}{9.8\ m/s^2}} = 1.7\ s$$

$$v_1 = \sqrt{2(y_0 - y_1)g} = \sqrt{2 \cdot 15\ m \cdot 9.8\ m/s^2} = 17\ m/s$$

$$a_2 = \frac{(y_0 - y_1)}{(y_1 - y_2)}g = \frac{15\ m}{1\ m}g = 15g$$

6. Let's evaluate our results

The dimensions and units are correct.

The free fall time seems right. For evaluating the "landing" speed we may use mph units

$$17\ m/s = 17\frac{m}{s}\left(\frac{1\ mile}{1609\ m}\right)\left(\frac{3600\ s}{1\ h}\right) = 38\ mph$$

It looks like a speed one would expect (fast and dangerous).

The deceleration is high $15g$ but the number is correct. It means that landing on a safety at given conditions is probably unpleasant experience.

3 Motion in Two Dimensions

Knowing the basics of two-dimensional motion will allow us to examine a wide variety of motions, ranging from a simple projectile motion to the motion of satellites, or orbit to the motion of electrons in a uniform electric field.

3.1 Position, displacement, velocity and acceleration in 2D and 3D

For motion in two or three dimensions we can extend the ideas from 1D motion for displacement, velocity and acceleration.

3.1.1 The displacement

Using the vector algebra we may define a *position vector* $\vec{r}$, which is a vector that extends from a reference point (usually the origin of a coordinate system) to the particle. In the unit-vector notation, it can be written

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{3.1}$$

where $x\hat{\imath}$, $y\hat{\jmath}$, and $z\hat{k}$ are the vector components of $\vec{r}$, and the coefficients x, y and z are its scalar components. The coefficients x, y and z give the particle's location along the coordinate axes and relative to the origin; that is, the particle has the rectangular coordinates (x, y, z).

Going from two to three dimension motion is just adding an additional coordinate z. For clarity we will mostly concentrate on 2D motion in (x, y) plane.

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin). If the position vector changes, say, from $\overrightarrow{r_1}$ to $\overrightarrow{r_2}$ during a certain time interval, then the particle's **displacement** $\Delta\vec{r}$ during that time interval is

$$\Delta\vec{r} = \overrightarrow{r_2} - \overrightarrow{r_1} \tag{3.2}$$

Using the unit-vector notation, we can rewrite this displacement as

$$\Delta\vec{r} = (x_2\hat{\imath} + y_2\hat{\jmath}) - (x_1\hat{\imath} + y_1\hat{\jmath})$$

or as

$$\Delta\vec{r} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} \tag{3.3}$$

where coordinates (x_1, y_1) correspond to position vector $\overrightarrow{r_1}$ and coordinates (x_2, y_2) correspond to position vector $\overrightarrow{r_2}$. We can also rewrite the displacement by substituting $\Delta x = (x_2 - x_1)$ and $\Delta y = (y_2 - y_1)$.

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$$

3.1.2 The average velocity and instantaneous velocity

If a particle moves through a displacement $\Delta\vec{r}$ in a time interval Δt, then its average velocity $\vec{v}_{avg}$ is

$$\vec{v}_{avg} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}.$$

This tells us the direction of $\vec{v}_{avg}$ must be the same as that of $\Delta\vec{r}$. Using the component form we can write

$$\vec{v}_{avg} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} \tag{3.4}$$

The instantaneous velocity $\vec{v}$ is defined as the limit of the average velocity

$$\vec{v} = \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}. \tag{3.5}$$

That is, the instantaneous velocity equals the derivative of the position vector with respect to time. The direction of the instantaneous velocity vector at any point in a particle’s path is along a line tangent to the path at that point and in the direction of motion.

In unit-vector form

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{\imath} + y\hat{\jmath}) = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} \tag{3.6}$$

or

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} \tag{3.7}$$

where the scalar components of $\vec{v}$ are

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}. \tag{3.8}$$

3.1.3 Average acceleration and instantaneous acceleration

When a particle's velocity changes from $\vec{v}_1$ to $\vec{v}_2$in a time interval Δt, its average acceleration during Δt is

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta \vec{v}}{\Delta t} \tag{3.9}$$

The instantaneous acceleration a is defined as the limiting value of the ratio $\Delta\vec{v}/\Delta t$, or the instantaneous acceleration equals the derivative of the velocity vector with respect to time.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}. \tag{3.10}$$

Important: If the velocity changes in either magnitude or direction (or both), the particle must have an acceleration.

In the unit vector notation

$$\vec{a} = \frac{d}{dt}\left(v_x\hat{\imath} + v_y\hat{\jmath}\right) = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x\hat{\imath} + a_y\hat{\jmath} \tag{3.11}$$

where

$$a_x = \frac{dv_x}{dt}, \qquad a_y = \frac{dv_y}{dt}$$

Thus, we can find the scalar components of $\vec{a}$ by differentiating the scalar components of $\vec{v}$.

3.2 Motion with constant acceleration in 2D

Let's consider a case when acceleration in a plane is constant. In 2D Cartesian coordinates

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = const_1\hat{\imath} + const_2\hat{\jmath}$$

Since in terms of velocity

$$\frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x\hat{\imath} + a_y\hat{\jmath}$$

then

$$\frac{dv_x}{dt} = a_x = const_1$$

$$\frac{dv_y}{dt} = a_y = const_2$$

Then we see that motion in x-direction is independent from motion in y-direction (but the time is still a common parameter).

Important: Very many standard university physics textbooks claim that "horizontal and vertical motions are independent". Generally it is not true. It is only correct for a set of very special cases. Motion with constant acceleration along both x and y coordinates is one of those cases.

The equations derived in Chapter 2 for motion in one dimension with constant acceleration can be applied separately to each of the perpendicular component of two-dimensional motion. If we let $\vec{v}_0 = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ be the initial velocity, then for the x and y components of the position

$$\begin{aligned} x &= x_0 + v_{x0}t + \frac{a_x t^2}{2} \\ y &= y_0 + v_{y0}t + \frac{a_y t^2}{2} \end{aligned} \tag{3.12}$$

and for their velocities

$$\begin{aligned} v_x &= v_{x0} + a_x t \\ v_y &= v_{y0} + a_y t \end{aligned} \tag{3.13}$$

The component form of the equations for the position (3.12) and for the velocity (3.13) show us that two-dimensional motion at constant acceleration is equivalent to two *independent* motions—one in the x direction and one in the y direction – having constant accelerations a_x and a_y.

The equations for components can be rewritten in the vector form as

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{\vec{a}t^2}{2} \tag{3.14}$$

$$\vec{v} = \vec{v}_0 + \vec{a}t \tag{3.15}$$

In practical situation we usually use the component forms (equations (3.12) and (3.13)).

To be as general as possible we could consider three-dimensional motion. However, in many interesting situations, a lot of the interesting physics can be studied using only two dimensions.

3.3 Projectile motion

We will limit our consideration to a simple projectile motion in xy plane due to free-fall acceleration neglecting the effects of air resistance and wind (that generally may affect both horizontal motion as well as vertical motion). Thus with these two assumptions

1. The free-fall acceleration g is constant over the range of motion and is directed downward (This assumption is reasonable as long as the range of motion is small compared with the radius of the Earth. In effect, this assumption is equivalent to assuming that the Earth is flat over the range of motion considered).
2. The effect of air resistance is negligible. (This assumption is generally not justified, especially at high velocities and will be discussed later. However, for very many types of projectiles moving with reasonable speeds the effect of air resistance is small)

3.3.1 Kinematic equations for simple projectile motion

Because air resistance is neglected, we know that $a_y = -g$ (as in one-dimensional free fall) and that $a_x = 0$. Furthermore, let us assume that at $t = 0$, the projectile leaves the origin (x_0, y_0) with speed v_0, as shown in Figure 20. The vector $\vec{v}_0$ makes an angle θ_0 with the horizontal.

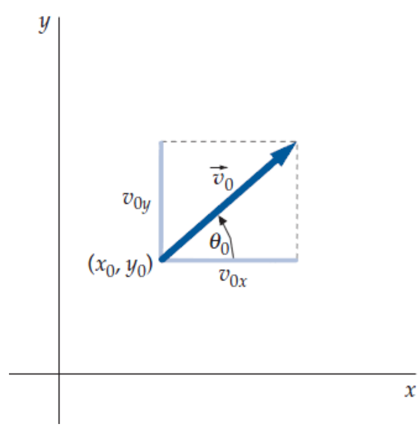


Figure 20 The components of $\vec{v}_0$ are v_{0x} and v_{0y}.

Therefore, the initial x and y components of velocity $\vec{v}_0$ are

$$v_{0x} = v_0 \cos\theta_0 \tag{3.16}$$

$$v_{0y} = v_0 \sin\theta_0 \tag{3.17}$$

Substituting $a_x = 0$ and $a_y = -g$ into (3.12) we find that the x component of the velocity is constant because no horizontal acceleration exists:

$$v_x = v_{0x} \tag{3.18}$$

The y component of the velocity varies with time according to

$$v_y(t) = v_{y0} - gt \tag{3.19}$$

Let's note again that v_x does not depend on v_y and v_y does not depend on v_x: The horizontal and vertical components of projectile motion are independent.

According to equation (3.13), the displacements x and y are given by

$$x = x_0 + v_{x0}t$$
$$y = y_0 + v_{y0}t - \frac{a_y t^2}{2}. \tag{3.20}$$

The notation $x(t)$ and $y(t)$ simply emphasizes that x and y are functions of time.

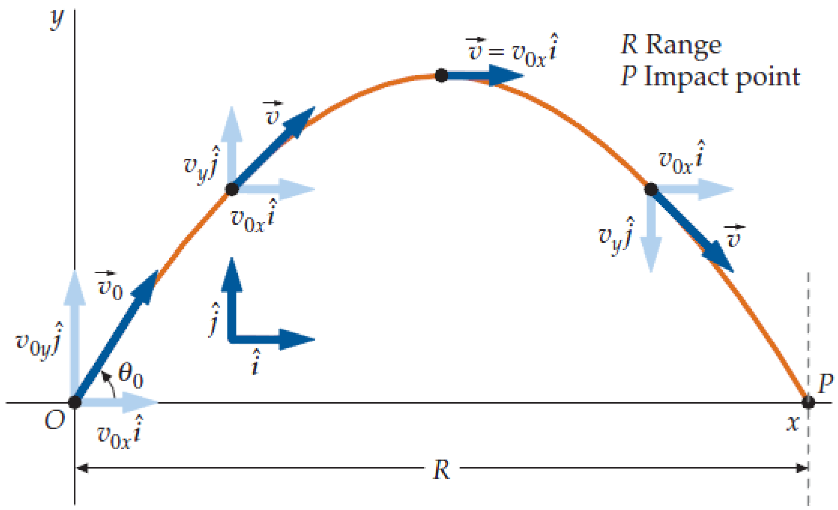


Figure 21 The path of a projectile, showing velocity components at different times.

Equations ((3.18) - (3.20)) form a complete set of equations for solving any simple projectile motion problem.

The general equation for the path $y(x)$ of a projectile can be obtained from equations (3.20) by eliminating the variable t

$$t = \frac{x - x_0}{v_{x0}}$$

that gives

$$y - y_0 = v_{y0}\left(\frac{x - x_0}{v_{x0}}\right) - \frac{1}{2}g\left(\frac{x - x_0}{v_{x0}}\right)^2$$

or after some rearrangement

$$y - y_0 = \frac{v_{y0}}{v_{x0}}(x - x_0) - \frac{1}{2}\frac{g}{v_{x0}^2}(x - x_0)^2. \tag{3.21}$$

This equation is of the form $y = ax + bx^2$, which is the equation for a parabola.

Using ((3.16) and (3.17)) we may rewrite ((3.21)) as

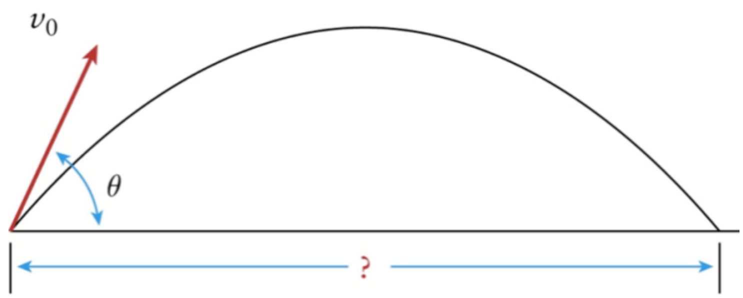
$$y - y_0 = \tan(\theta)(x - x_0) - \frac{g(x - x_0)^2}{2v_0^2\cos^2\theta} \tag{3.22}$$

Equation ((3.22)) is useful enough when t (time) is no interest.

3.3.2 A couple interesting cases

Case 1: Horizontal range of a projectile on a flat surface (How far?)

You toss a ball into the air with initial speed v_0 and at initial angle θ from the horizontal. (For simplicity we will use θ instead of θ_0 for the initial angle). Neglecting any effect due to air resistance, how far has the ball travelled horizontally when it returns to the *initial* launch height?



From the kinematic equations of motion follows

for the position for the velocity

$$x = x_0 + v_0 \cos(\theta)\, t \qquad v_x = v_0 \cos(\theta)$$

$$y = y_0 + v_0 \sin(\theta)\, t - \frac{gt^2}{2} \qquad v_y = v_0 \sin(\theta) - gt$$

The condition "returns to the initial launch height" means $y = y_0$, then equation for y

$$0 = v_0 \sin(\theta)\, t_{total} - \frac{g\, t_{total}^2}{2},$$

where t_{total} denotes the time of flight for the ball to complete the flight

$$t_{total} = \frac{2v_0 \sin(\theta)}{g}. \tag{3.23}$$

When this specific time is substituted into equation for x

$$(x - x_0)_{range} = v_0 \cos(\theta) \frac{2v_0 \sin(\theta)}{g} = \frac{2v_0^2 \sin(\theta) \cos(\theta)}{g}$$

using the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ we can find the range

$$R = (x - x_0)_{range} = \frac{v_0^2 \sin(2\theta)}{g}. \tag{3.24}$$

By the way, solving for the range we also got the total time of flight t_{total} (equation (3.23)).

Let's also find the velocity at the ground. Using the time of flight we may find the vertical component of the velocity at the impact

$$v_y = v_0 \sin(\theta) - g \frac{2v_0 \sin(\theta)}{g} = -v_0 \sin(\theta)$$

Then the total impact speed is

$$v_{impact} = \sqrt{(v_0\cos(\theta))^2 + (-v_0\sin(\theta))^2} = v_0 \tag{3.25}$$

The impact speed is the same as the initial speed.

Let's find out what angle provides the largest range if the ball returns to the initial launch height? Assume that the initial position $x_0 = 0$ then

$$R = \frac{v_0^2\sin(2\theta)}{g}. \tag{3.26}$$

The largest range corresponds

$$\frac{dR}{d\theta} = 2\frac{v_0^2}{g}\cos(2\theta) = 0.$$

The minimum value of $\cos(2\theta) = 0$ occurs when $2\theta = 90^0$. Therefore, R is a maximum when $\theta = 45^0$. Figure 22 illustrates various trajectories for a projectile having a given initial speed but launched at different angles. As you can see, the range is a maximum for $\theta = 45^0$. In addition, for any θ other than 45^0, a point having cartesian coordinates $(R, 0)$ can be reached by using either one of two complementary values of θ, such as 75° and 15°. Of course, the maximum height and time of flight for one of these values of θ are different from the time of flight for the complementary value.

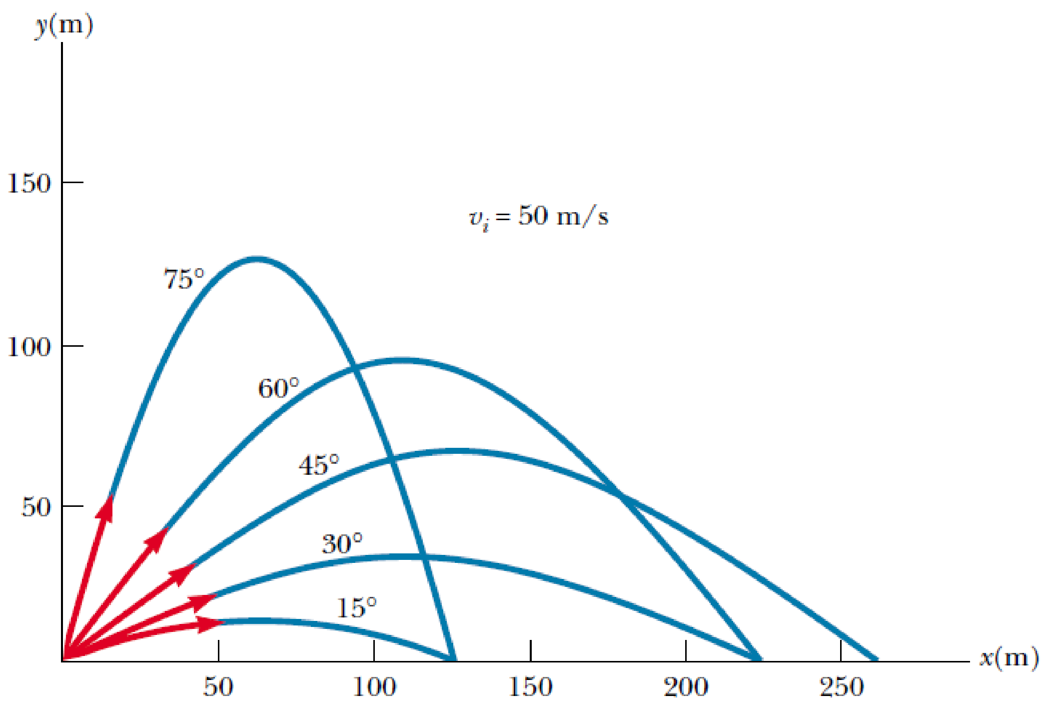
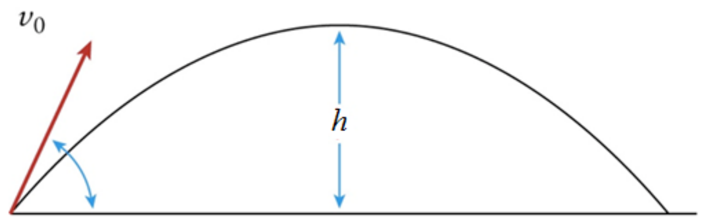


Figure 22 A projectile fired from the origin with an initial speed of 50 m/s at various angles of projection. Note that complementary values of θ result in the same value of x (range).

Case 2: Maximum Height of a Projectile (How high?)

You toss a ball into the air with initial speed v_0 and at initial angle θ from the horizontal. Neglecting any effect due to air resistance, how high does the ball go before coming back down?



The kinematic equations of motion (again)

for the position for the velocity

$$x = x_0 + v_0 \cos(\theta)\, t \qquad v_x = v_0 \cos(\theta)$$

$$y = y_0 + v_0 \sin(\theta)\, t - \frac{gt^2}{2} \qquad v_y = v_0 \sin(\theta) - gt$$

At the highest point on the trajectory $v_y = 0$ then from $0 = v_0 \sin(\theta) - gt$

$$t_{peak} = \frac{v_0 \sin(\theta)}{g} \tag{3.27}$$

Let's note that this time is half of the total time of flight (equation (3.23)). We can now substitute this time into equation for y

$$(y - y_0)_{peak} = v_0 \sin(\theta) \frac{v_0 \sin(\theta)}{g} - \frac{g}{2}\left(\frac{v_0 \sin(\theta)}{g}\right)^2$$

$$= \frac{v_0^2 \sin^2(\theta)}{g} - \frac{1}{2}\frac{v_0^2 \sin^2(\theta)}{g} = \frac{1}{2}\frac{v_0^2 \sin^2(\theta)}{g}$$

and finally

$$(y - y_0)_{peak} = h = \frac{1}{2}\frac{v_0^2 \sin^2(\theta)}{g}.$$

Assume that the initial position $y_0 = 0$ then

$$h = \frac{v_0^2 \sin^2(\theta)}{2g}. \tag{3.28}$$

What x-coordinate corresponds to the peak position? We can find that by using t_{peak} in $x = v_0 \cos(\theta) t$

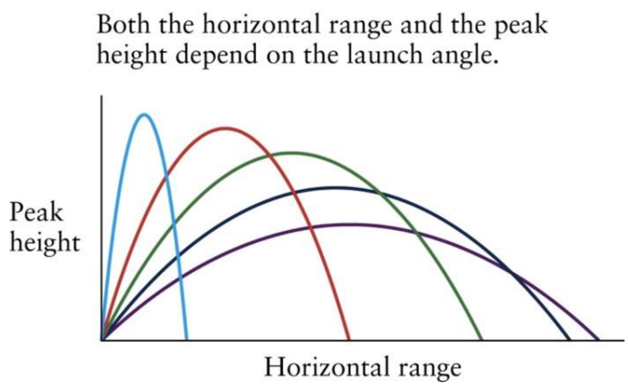
$$x = v_0 \cos(\theta) \frac{v_0 \sin(\theta)}{g} = \frac{1}{2}\frac{v_0^2 \sin(2\theta)}{g} = \frac{1}{2} R$$

where R is the range of a projectile. Thus, the highest point on the trajectory corresponds to the half of the range. The same results we can get by differentiating the trajectory (3.22)

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan(\theta)x - \frac{gx^2}{2v_0^2\cos^2\theta}\right) = \tan(\theta) - \frac{gx}{v_0^2\cos^2\theta} = 0$$

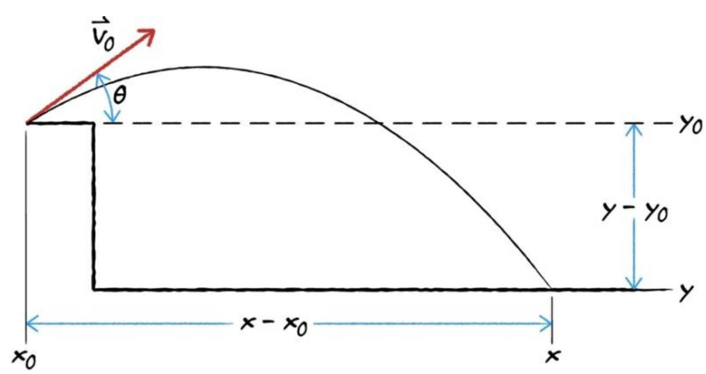
$$x = \frac{\sin(\theta)}{\cos(\theta)}\frac{v_0^2}{g}\cos^2\theta = \frac{1}{2}\frac{v_0^2\sin(2\theta)}{g} = \frac{1}{2}x_{range}$$

Note:



Case 3: Hitting a ball from a cliff

In very many problems an object is launched from one vertical position and lands at another, i.e. $y \neq y_0$. You hit a ball off cliff at an initial speed v_0 and at initial angle θ from the horizontal. How far from the base of the cliff does the ball travel before hitting the ground?



Now there is nothing to simplify in the original system of equations

$$x = x_0 + v_0\cos(\theta)\,t \qquad v_x = v_0\cos(\theta)$$

$$y = y_0 + v_0\sin(\theta)\,t - \frac{gt^2}{2} \qquad v_y = v_0\sin(\theta) - gt$$

We may solve the problem by finding solutions of the quadratic equation for t

$$y = y_0 + v_0\sin(\theta)\,t - \frac{gt^2}{2}$$

Having the time we can easily find the distance from $x = x_0 + v_0 \cos(\theta)\, t$.

Or we can derive equation for the trajectory (see equation (3.22))

$$y - y_0 = \tan(\theta)\,(x - x_0) - \frac{g(x - x_0)^2}{2v_0^2 \cos^2 \theta}.$$

Since the choice of the initial position is always ours to make, we can set $x_0 = 0$ and $y - y_0 = h$. Then rewriting the last equation for the trajectory

$$\frac{g}{2v_0^2 \cos^2 \theta} x^2 - (\tan \theta) x - h = 0.$$

This is a quadratic equation

$$ax^2 + bx + c = 0$$

The general solution to a quadratic equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, in this case the roots are

$$x = \frac{\tan \theta \pm \sqrt{(-\tan \theta)^2 - 4\left(\frac{g}{2v_0^2 \cos^2 \theta}\right)(-h)}}{2\left(\frac{g}{2v_0^2 \cos^2 \theta}\right)}.$$

There are two solutions of the quadratic equation. The larger value corresponds to the distance of interest. The smaller value describes a case if the ball was launched from the ground distance x_{back}"behind" the cliff.

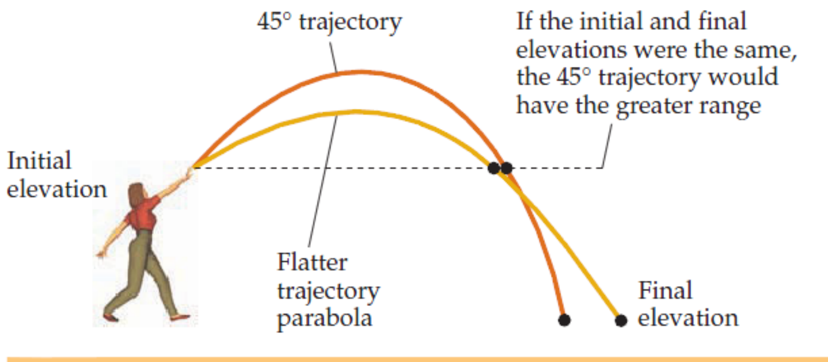
It is easy to show that for $h = 0$ this equation gives the range for $y = y_0$ namely our equation (3.26)

$$x = \frac{v_0^2 \sin(2\theta)}{g}.$$

Attention: If a projectile lands at an elevation lower than the initial elevation, the maximum horizontal displacement is achieved when the projection angle is different from 45^0. We can find the angle that maximize the range of a projectile launched with speed v_0 from height h above the ground by setting $dx/d\theta = 0$. After about a full page of of algebra and trigonometry the answer is

$$\theta_{max} = \frac{1}{2} \cos^{-1}\left(\frac{1}{1 + v_0^2/gh}\right).$$

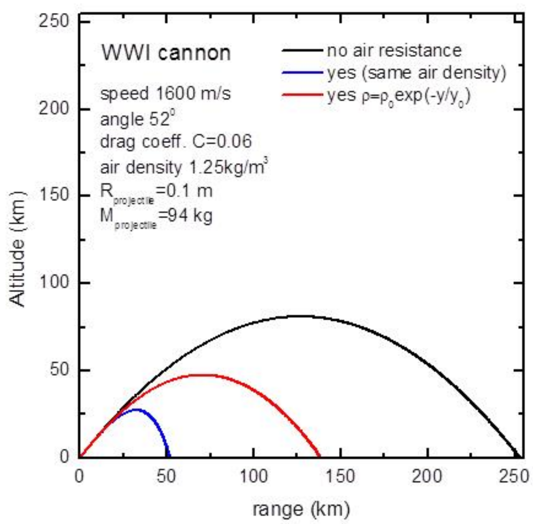
For $h > 0$ the angle θ_{max} is less than 45^0.

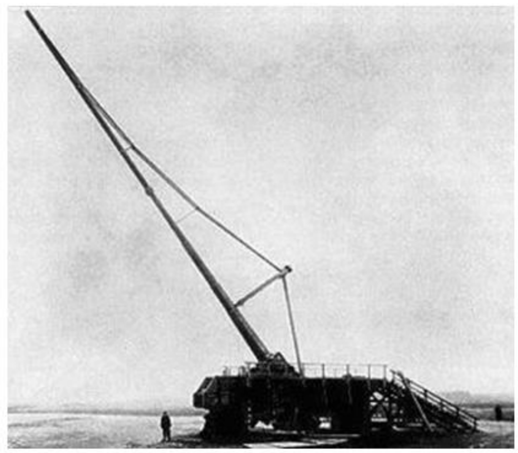

45° trajectory
If the initial and final elevations were the same, the 45° trajectory would have the greater range
Initial elevation
Flatter trajectory parabola
Final elevation

3.3.3 *Effects of air resistance for fast moving objects*

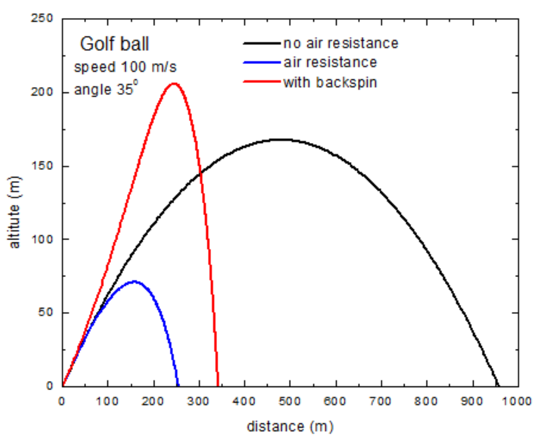
For *fast moving* objects the effect of air resistance can be very significant.

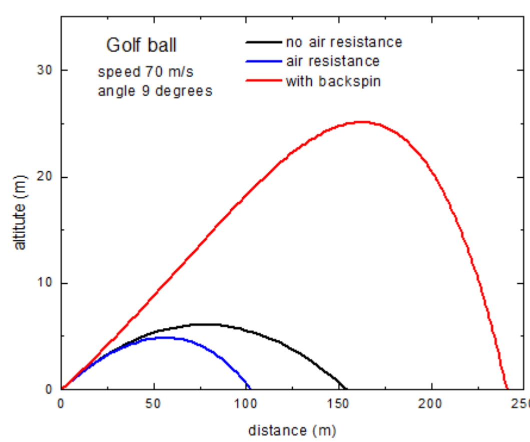
(a) the largest canon (Paris gun or *Paris-Geschütz*) in World War I (calculations by Alex G)





(b) a trajectory of a golf ball (calculations by Alex G)





(c) free-fall with air resistance

The acceleration in y direction is no longer a constant, its magnitude decreases as speed increases. When vertical acceleration reaches zero, an object reaches a terminal velocity.

object	speed (m/s)	speed (mph)	distance (m) 95%
shot	145	316	2500
sky diver	60	130	430
baseball	42	92	210
basketball	20	44	47
raindrop	7	15	6
parachutist	5	11	3

3.4 Motion in a circle

When a particle moves along a curved path, its velocity changes the direction. This means that the particle must have a component of acceleration perpendicular to the path, even if the speed is constant. Acceleration is defined as

$$\vec{a} = \frac{d\vec{v}}{dt},$$

where both acceleration and velocity are vectors. We can write $\vec{v} = v\hat{v}$ as a magnitude v multiplied by a direction $\hat{v}$. Since the derivative of a product is

$$\frac{d}{dt}(xy) = \frac{dx}{dt}y + \frac{dy}{dt}x$$

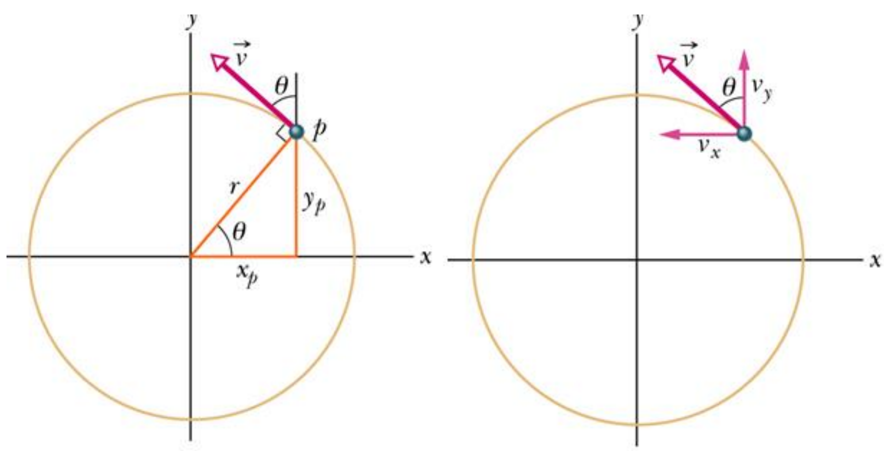
then for the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{v} + \frac{d\hat{v}}{dt}v \qquad (3.29)$$

The expression clearly shows that acceleration is nonzero when either speed dv/dt or direction $d\hat{v}/dt$ or both are changing.

3.4.1 A uniform circular motion

A particle is in **uniform circular motion** if it travels around a circle or a circular arc at constant (uniform) speed. To find the magnitude and direction of the acceleration for uniform circular motion we consider a particle moving at constant speed v around a circle of radius r.



Recall that velocity $\vec{v}$ of a moving particle is always tangent to the particle's path at the particle's position. It means $\vec{v}$ is perpendicular to the radius $\vec{r}$. Then, as one can see from figure above, the angle θ that $\vec{v}$ make with the vertical at p equals the angle θ that radius r makes with the x axis.

The velocity in the component form can be written as

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (-v\sin\theta)\hat{\imath} + (v\cos\theta)\hat{\jmath}.$$

Now we replace $\sin\theta$ with y_p/r and $\cos\theta$ with x_p/r thus getting

$$\vec{v} = \left(-v\frac{y_p}{r}\right)\hat{\imath} + \left(v\frac{x_p}{r}\right)\hat{\jmath}$$

Consequently the acceleration is (remember that we use v is a constant)

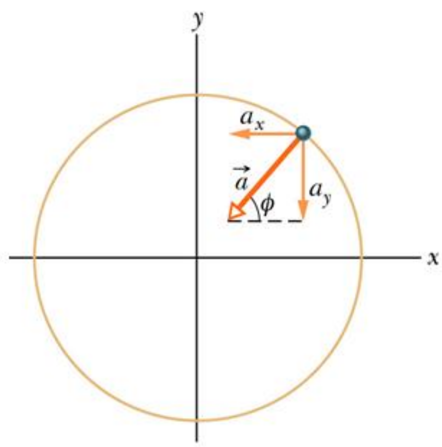
$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_p}{dt}\right)\hat{\imath} + \left(\frac{v}{r}\frac{dx_p}{dt}\right)\hat{\jmath}$$

but

$$\frac{dy_p}{dt} = v_y = v\cos\theta, \qquad \frac{dx_p}{dt} = v_x = -v\sin\theta$$

then finally

$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{\imath} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{\jmath}.$$



The magnitude of the acceleration is

$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \frac{v^2}{r} \tag{3.30}$$

which is constant! Such acceleration is called centripetal acceleration.

Let's analyze the direction of the acceleration.

$$\tan\phi = \frac{a_y}{a_x} = \frac{-\frac{v^2}{r}\sin\theta}{-\frac{v^2}{r}\cos\theta} = \tan\theta$$

Thus, $\phi = \theta$ which means that $\vec{a}$ is directed along the radius r *toward the circle's center*. Finally, the acceleration of an object moving in uniform circular motion (centripetal acceleration) in vector form

$$\vec{a} = \frac{v^2}{r}\hat{r} \tag{3.31}$$

In addition, during this acceleration (it happens at constant speed!), the particle travels the circumference (a distance of $2\pi r$) in time

$$T = \frac{2\pi r}{v} \tag{3.32}$$

3.4.2 A nonuniform circular motion

In a general case of a *nonuniform* circular motion (speed is not a constant) there are two components of acceleration, namely $a_\perp$ and $a_\parallel$

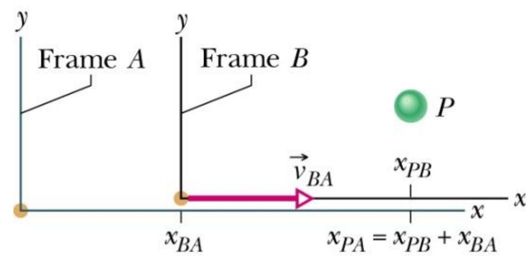
$$a_\perp = \frac{v^2}{r} a_\parallel = \frac{d|\vec{v}|}{dt}$$

We will talk about that in later chapters (Kinematics and dynamics of rotational motion).

3.5 Relative motion in one and two dimensions

Suppose two cars are moving in the same direction with speeds of $50\,mph$ and $60\,mph$. To a passenger in the slower car, the speed of the faster car is $10\,mph$. Of course, a stationary observer will measure the speed of the faster car to be $60\,mph$, not $10\,mph$. Which observer is correct? They both are! This simple example demonstrates that the velocity of an object depends on the frame of reference in which it is measured.

Let's first consider a relative motion in one dimension with a car A (stationary), a car B (moving relative to A with *constant* speed v_{BA}) and a ball P (moving with constant speed v_{PB} relative to B).



What are the ball's position and speed relative to A? In this case it is clear that

$$x_{PA} = x_{PB} + x_{BA}$$

Differentiating we get

$$\frac{d}{dt}x_{PA} = \frac{d}{dt}x_{PB} + \frac{d}{dt}x_{BA}$$

or $v_{PA} = v_{PB} + v_{BA}$. The acceleration

$$\frac{d}{dt}v_{PA} = \frac{d}{dt}v_{PB} + \frac{d}{dt}v_{BA}$$

In a general case

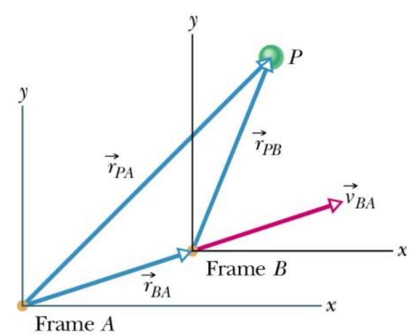
$$a_{PA} = a_{PB} + a_{BA}$$

however for $v_{BA} = const$ we have

$$a_{PA} = a_{PB}$$

Thus the velocity of a particle depends on a reference frame. But observers on different reference frames (that move at constant velocity relative to each other) will measure the same acceleration for a moving particle.

We can easily do the same for relative motion in two dimensions



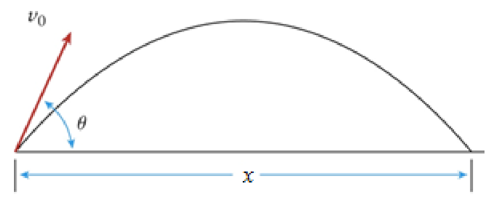
showing that

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$
$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$
$$\vec{a}_{PA} = \vec{a}_{PB}$$

As for one-dimensional motion, observers on different frames of reference that that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

3.6 Most common problems involving projectile motion

Type 1: "the same ground level" or $y = y_0$.



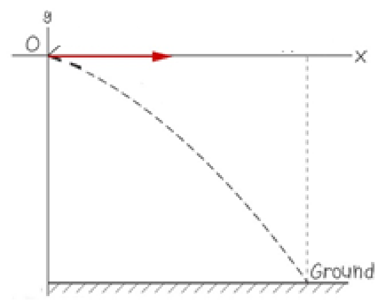
This is rather a simple case because the time of flight can easily be found from

$$0 = v_0 \sin(\theta)\, t_{flight} - \frac{g t_{flight}^2}{2}$$

Then you have simple algebra calculations.

The highest point on the trajectory corresponds to $v_y = v_0 \sin(\theta) - gt = 0$

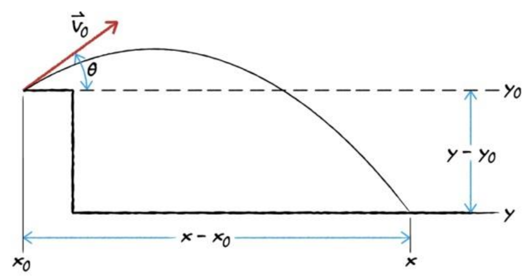
Type 2: "zero horizontal launch" or $\theta = 0^0$.



This is the easiest type of projectile motion problems with very easy to solve algebra for the system of equations

$$x = v_0 t \qquad v_x = v_0$$
$$0 = y_0 - \frac{gt^2}{2} v_y = -gt$$

Type 3: "a general case" with a nonzero launch angle and when $y > y_0$ or $y < y_0$.



Quite often you may need to solve a quadratic equation if you are looking for time or distance. For example, depending on the given data, the time of flight can be found right away from

$$y = y_0 + v_0 \sin(\theta)\, t - \frac{gt^2}{2}$$

If finding v_0 is your goal, then the problem can be reduced to simple algebra, but if you need to find the initial angle θ, you may need to be engaged into trigonometry.

3.7 Examples

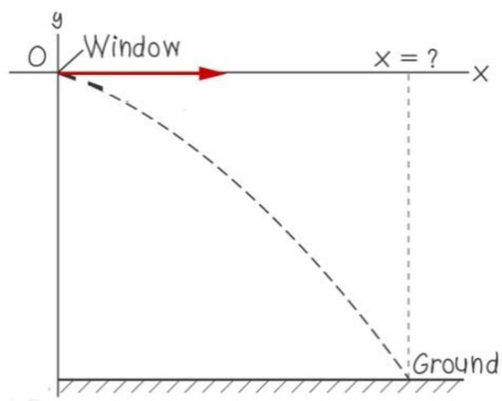
Example 3-1

A 10 pound cat leaps horizontally from a 3.1 m – high window with a speed of 5.0 m/s. Disregard the air resistance.

a) How far from the base of the house will she land?
b) How long will be the flight time?

SOLUTION:

1. Physics – projectile motion in a plane



2. The basic equations for projectile motion in a plane (note that the mass of the cat does not matter!)

$$x = x_0 + v_0\cos(\theta)\,t \qquad v_x = v_0\cos(\theta)$$

$$y = y_0 + v_0\sin(\theta)\,t - \frac{gt^2}{2} \qquad v_y = v_0\sin(\theta) - gt$$

3. Using given data $\theta = 0^0$ and $x_0 = 0,\ y = 0$ the equation above can be written in the following form

$$x = v_0 t \quad (1x) \qquad v_x = v_0 \quad (2x)$$

$$0 = y_0 - \frac{gt^2}{2} (1y) \qquad v_y = -gt \quad (2y)$$

4. There are two unknowns in equations $(1x)$ and $(1y)$, namely $x$and t. The first equation $(1x)$ has two unknowns but the second has only one unknown – the time

Solving $(1y)$ for time and using it in $(1x)$ gives

$$t_{flight} = \sqrt{\frac{2y_0}{g}} x_{distance} = v_0\sqrt{\frac{2y_0}{g}}$$

5. Calculations:

$$t_{flight} = \sqrt{\frac{2 \cdot 3.1\ m}{9.8\ m/s^2}} = 0.8\ s \qquad x_{distance} = 5.0\ m/s \cdot 0.8\ s\ = 4\ m$$

6. Let's evaluate the results

The dimensions and units are correct. The numerical results look realistic for a regular cat.

Example 3-2

In The Dukes of Hazzard (2005), a 1969 Dodge Charger (3256.0 lbs) went 175.0 ft after taking off from a ramp inclined at 30^0 degrees. In the movie the ramp was about 6.0 ft tall.

a) How fast should the car be traveling (in mph) at the end of the ramp to make the 175 ft jump (counting from the end of the ramp)?
b) How much time would the jump take?
c) How high would be the highest point of the trajectory?

SOLUTION:

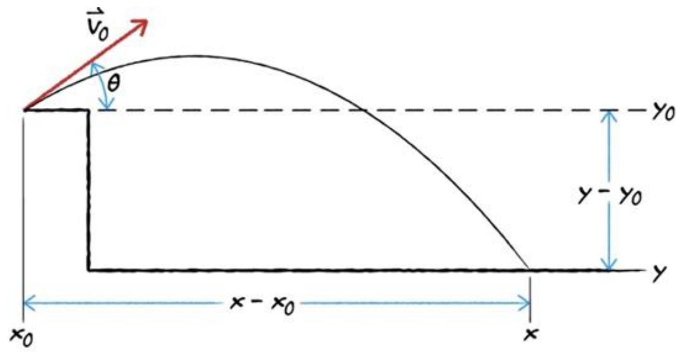
1. Physics – projectile motion

2. The basic equations for projectile motion in a plane

$$x = x_0 + v_0 \cos(\theta)\, t \qquad\qquad v_x = v_0 \cos(\theta)$$

$$y = y_0 + v_0 \sin(\theta)\, t - \frac{gt^2}{2} \qquad\qquad v_y = v_0 \sin(\theta) - gt$$

3. The problem is quite general with nothing to simplify



4. Let's analyze the number of unknowns in every equation. for the positions x and y. The both equations have the same two unknowns, namely $v_0$and t. From the first equation

$$t = \frac{x - x_0}{v_0 \cos\theta}$$

Substituting it into the second equation gives

$$y - y_0 = v_0 \sin\theta \frac{x - x_0}{v_0 \cos\theta} - \frac{g(x - x_0)^2}{2v_0^2 \cos^2\theta}.$$

This equations has only one unknown v_0 that can be easily found

$$v_0^2 = \frac{1}{2} \frac{g(x - x_0)^2}{[(x - x_0)\tan\theta - (y - y_0)]\cos^2\theta}.$$

Let's test our solution in case when $y - y_0 = 0$ (disregard the size of the ramp). From the equation above we can derive

$$v_0^2 = \frac{1}{2} \frac{g(x - x_0)^2}{(x - x_0)\sin\theta\cos\theta} = \frac{g(x - x_0)}{\sin 2\theta}$$

that is the equation for the horizontal range on a flat surface (correct). Now we have the speed and the time. The highest point on the trajectory can be found from the condition $v_y = 0$ at this point. So

$$0 = v_0 \sin(\theta) - gt$$

and $t = v_0 \sin(\theta)/g$. Then we use this time in the equation for y.

5. Calculations

The initial data in SI units (we use 1 ft = 0.3048 m)

$$175\,ft = 175\,ft\left(\frac{0.3048\,m}{1\,ft}\right) = 53.3\,m 6\,ft = 6\,ft\left(\frac{0.3048\,m}{1\,ft}\right) = 1.8\,m$$

The results are The speed $v_0 = 24.6\ m/s$

$a) v_0 = 24\ m/s = 53\,mph \quad b)\ t = 2.6\,s \quad c)\ h = 9.1\,m$

6. All units and dimensions are correct. The numbers seem realistic.

Example 3-3

You throw an apple from the upper edge of 220-m vertical dam with a speed of 25.0 m/s at 30.0^0 above the horizon. Ignore air resistance.

a) How long after throwing the apple will you see it?
b) How far from the base of the dam will the apple strike the water surface?
c) What will be the speed of the apple when entering water

SOLUTION:

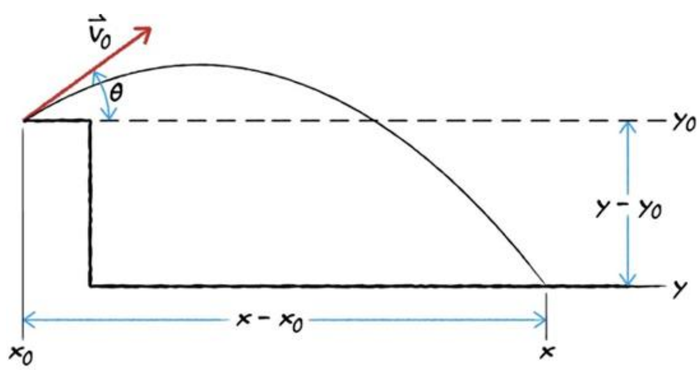
1. Physics – projectile motion

2. The basic equations for projectile motion in a plane

$$x = x_0 + v_0 \cos(\theta)\, t \qquad v_x = v_0 \cos(\theta)$$

$$y = y_0 + v_0 \sin(\theta)\, t - \frac{gt^2}{2} \qquad v_y = v_0 \sin(\theta) - gt$$

3. The problem is quite general with nothing to simplify



4. Let's analyze the number of unknowns in every equation. The first equation has two unknowns, namely distance x and time t, and the second equation has one unknown (time t). Thus we can solve the second (quadratic) equation for time that looks like $at^2 + bt + c = 0$ where $a = -g/2$, $b = v_0 \sin(\theta)$, $c = y_0 - y$. The rots are

$$t = \frac{-v_0 \sin(\theta) \pm \sqrt{v_0^2 \sin^2(\theta) + 2g(y_0 - y)}}{-g}$$

then $x = x_0 + v_0 \cos(\theta)\, t$ will give the distance.

The final speed is $v = \sqrt{v_x^2 + v_y^2}$ where $v_x = v_0 \cos(\theta)$ and $v_y = v_0 \sin(\theta) - gt$

5. Calculations

The quadratic equation for the time has two roots, namely $t = -5.55\ s$, and $t = 8.10\ s$. The positive solution corresponds to the time we are looking for (do you know why? What does the negative solution mean?).

The distance $x = 0.0\, m + 25\ m/s * \cos 30^0 * 8.10\, s = 175\, m$

The speed $v = 70.3\ m/s$

$a)\ t = 8.10\, s \quad b)\ x = 175\, m \quad c)\ v = 70.3\, m/s$

6. All units and dimensions are correct. The numbers seem realistic.

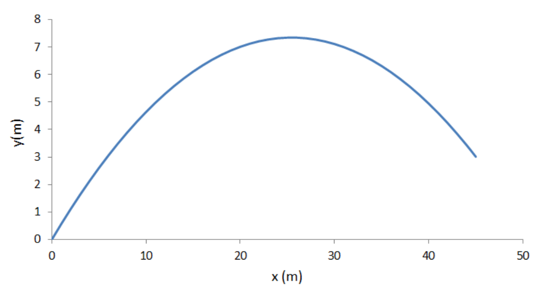
Example 3-4

The Chickens are playing with the Turkeys in football and the score is tied. A placekicker is sent out for the Chickens with instructions from the coach to kick a field goal from 45.0 m out. The top of the cross bar on the goalpost is 3.05 m above the level playing field. The moment of truth arrives and the ball leaves the ground at an angle of 30.0^0 to the horizontal.

a) What is the minimum speed that the ball must have to make the field goal?
b) How long does it take the ball to reach the cross bar?

SOLUTION:

1. Physics – projectile motion in a plane



2. The basic equations for projectile motion in a plane

$$x = x_0 + v_0\cos(\theta)\, t \qquad v_x = v_0\cos(\theta)$$

$$y = y_0 + v_0\sin(\theta)\, t - \frac{gt^2}{2} \qquad v_y = v_0\sin(\theta) - gt$$

3. It looks like we have to deal with the basic equations without simplifying them.

We can only set $x_0 = 0\, m$ and $y_0 = 0\, m$.

4. Let's work with equations for x and y. These two equations have two unknowns, namely t and v_0.

From the first equation

$$t = \frac{x - x_0}{v_0\cos(\theta)}$$

then the second equation reads

$$y = y_0 + v_0\sin(\theta)\frac{x - x_0}{v_0\cos(\theta)} - \frac{g}{2}\left(\frac{x - x_0}{v_0\cos(\theta)}\right)^2 = y_0 + \sin(\theta)\frac{x - x_0}{\cos(\theta)} - \frac{g}{2}\frac{(x - x_0)^2}{v_0^2\cos^2(\theta)}$$

and after some algebra

$$v_0^2 = \frac{1}{2}\frac{g(x - x_0)^2}{[(x - x_0)\tan\theta - (y - y_0)]\cos^2\theta}$$

5. Calculations:

$$v_0^2 = 0.5 \frac{9.8\ m/s^2 \cdot\ (45.0\,m)^2}{[45\,m\ \cdot \tan 30^0 - (3.05\,m - 0\,m)] \cos^2 30^0} = 577\ m^2/s^2\ v_0 = 24.0\ m/s$$

$$t = 2.16\,s$$

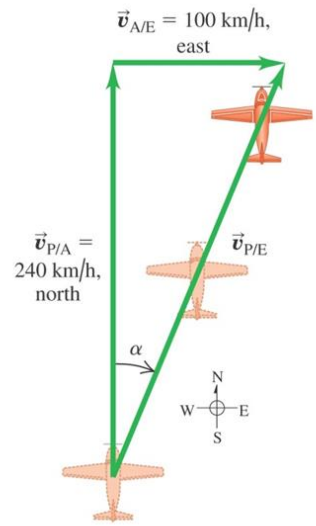
6. Looking back: The dimensions and units are correct. The speed (24 m/s or 54 mph) looks challenging but reasonable. The time seems correct.

Example 3-5*

Flying in crosswind

The compass of an airplane indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at speed 240 km/h. If there is a wind of 100 km/h from west to east,

a) What is the velocity of the airplane relative to the ground?



SOLUTION:

1. Physics – Relative motion in a plane with constant velocities

2. Equations

$$\vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE}$$

3. and 4. We can find the solution immediately from the equation above using geometry of vector components. Let's use vector components as the most general approach

$\vec{v}_{PA} = 0\hat{x} + v_{PA}\hat{y}\vec{v}_{AE}, \;\; \vec{v}_{AE} = v_{AE}\hat{x} + 0\hat{y},$

$\vec{v}_{PE} = (0 + v_{AE})\hat{x} + (v_{PA} + 0)\hat{y}$

$v_{PE} = \sqrt{(0 + v_{AE})^2 + (v_{PA} + 0)^2}, \qquad \alpha = \arctan v_{AE}/v_{PA}$

Here we use $\hat{x}$ and $\hat{y}$ as notations for unit vectors in x- and y-directions.

5. Calculations

Since we are not asked about specific units of velocity, we may proceed with km/h

$v_{PE} = \sqrt{(240\ km/h)^2\ + (100\ km/h)^2} = 260\ km/h$

$\alpha = \arctan v_{AE}/v_{PA} = \arctan 100/240 = 23^0$

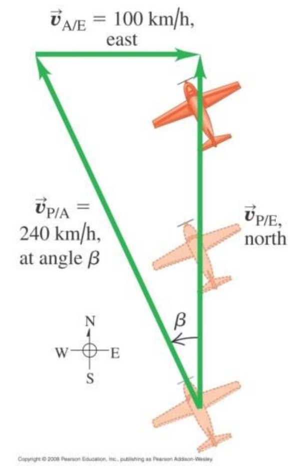
6. The dimensions and units look right (it is rather unlikely to make an error with this simple problem). The numerical results seem reasonable.

Example 3-6*

Flying in crosswind II

An airspeed indicator shows that an airplane is moving through the air at speed 240 km/h. If there is a wind of 100 km/h from west to east

a) In what direction should the pilot head to travel due north.
b) What will be his velocity relative to the earth?



SOLUTION:

1. Physics – Relative motion in a plane with constant velocities

2. Equations

$$\vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE}$$

3. and 4. We can find the solution using vector components. Let's use vector components as the most general approach

$\vec{v}_{PA} = -v_{PA}\sin\beta\,\hat{x} + v_{PA}\cos\beta\,\hat{y}$ $\quad\quad$ $\vec{v}_{AE} = v_{AE}\hat{x} + 0\hat{y}$,

$\vec{v}_{PE} = (v_{AE} - v_{PA}\sin\beta)\hat{x} + (v_{PA}\cos\beta + 0)\hat{y}$

$\sin\beta = v_{AE}/\,v_{PA}$

Here we use $\hat{x}$ and $\hat{y}$ as notations for unit vectors in x- and y-directions.

By the way, using equation for the angle β iy is easy to show that

$$v_{PE} = v_{PA}\cos\beta = \sqrt{v_{PA}^2 - v_{AE}^2}$$

5. Calculations

$v_{PE} = \sqrt{(240\ \ km/h)^2 - (100\ \ km/h)^2} = 218\,km/h$

$\beta = \arcsin 100/240 = 25^0$

6. The dimensions and units look right (it is rather unlikely to make an error with this simple problem). The numerical results seem reasonable

4 Newton’s Laws of Motion

4.1 Dynamics

Kinematics equations of motion (specifically motion with constant acceleration) are essentially mathematical equations. The equations do not have any physics or causes of motion.

Dynamics of motion describes motion together with its cause. The discovery of the laws of dynamics, or the laws of motion, was a dramatic moment in the history of science. Before Sir Isaac Newton's time (1642 – 1727), the motions of things like the planets were a mystery, but after Newton there was complete understanding. The motions of objects around us (from a grain of sand to stars and planets), could all be analyzed completely after Newton's laws were enunciated.

Galileo made a great advance in the understanding of motion when he discovered the principle of inertia: if an object is left alone, or not disturbed, it continues to move with a constant velocity in a straight line if it was originally moving, or it continues to stand still if it was just standing still (looks counterintuitive, isn’t it). It required a certain imagination to find the right rule, and that imagination was supplied by Galileo.

Of course, the next thing which is needed is a rule for finding how an object changes its speed if something is affecting it. That is the contribution of Newton. Newton wrote down three laws (published in 1687): The First Law was a mere restatement of the Galilean principle of inertia just described. The Second Law gave a specific way of determining how the velocity changes under different influences called forces. The Third Law is a relationship between the forces that two interacting bodies exert on each other.

Newton's laws are truly fundamental, for they cannot be deduced or proved from other principles. Newton's laws are based on observations made by many scientists before Newton.

Newton's laws are the foundation of classical mechanics.

There are two situations when Newtonian mechanics cannot be applied, namely if the speed of interacting bodies is comparable with the speed of light (then we use Einstein's special theory of relativity), and if the interacting bodies are comparable with (or smaller than) atomic scale (then we use quantum mechanics). We can view Newtonian mechanics as a special case of the two above fundamental theories. However, Newtonian mechanics can be applied to vast majority of situations, especially on our Human scale.

4.2 Force and Interaction

Although it is interesting and worthwhile to study the physical laws simply because they help us to understand and to use nature, one ought to stop every once in a while and think, "What do they really mean?" Well, we can intuitively sense the meaning of force. In physics, the answer is simple: "If a body is accelerating, then there is a force on it."

Since forces always come in pairs we can also say that a *force* is an interaction between two bodies or a body and its environment.

In nature, there are *three only fundamental forces* (gravity, electroweak force and strong force). All three forces originate from long-distance interactions at microscopic level.

Since the electromagnetic part of the weak part of the electroweak force act on different scales it is convenient for multiple applications to consider them as two distinct forces.

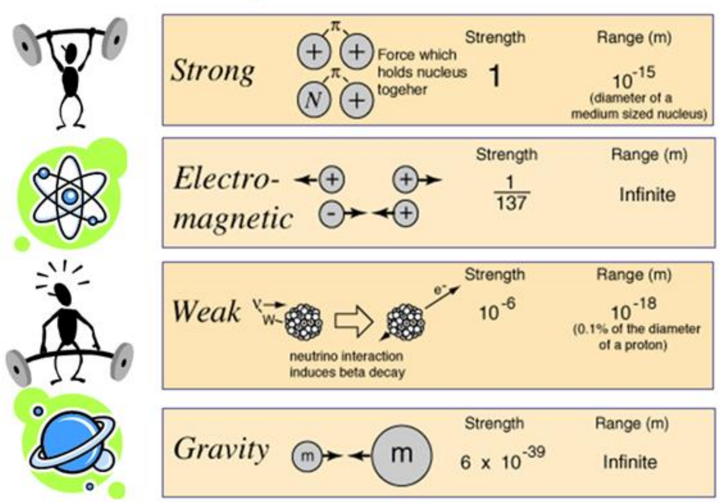


Figure 23 Relative strength and the range of interaction of the four fundamental forces.

Many scientists think that all the fundamental forces are the manifestations of a single force which has yet to be discovered.

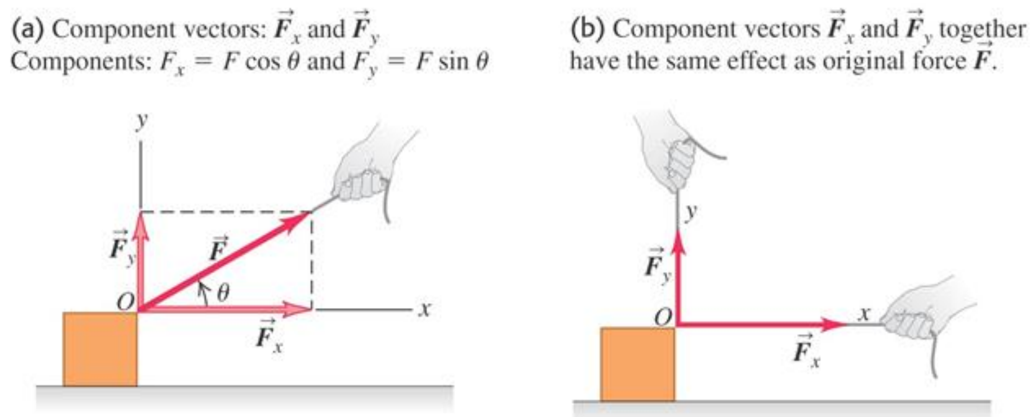
Quick comments: in many textbook one may find multiple definitions for a force. As a rule, most definitions are useless like a discussion what definition of a word is better. Also it is common to talk about contact forces and long-range forces. A contact force is a simplified model of reality.

Important. Force is a vector $\vec{F}$.

In University physics we often use the Cartesian coordinate system. Then, in two dimensional case

$$\vec{F} = \vec{F}_x + \vec{F}_y = F_x\hat{x} + F_y\hat{y} \tag{4.1}$$

where $\vec{F}_x$ and $\vec{F}_y$ are component vectors along $x-$ and $y-$directions, and $\hat{x}$ and $\hat{y}$ are unit vectors.



Any number of forces applied at a point on a body has the same effect as a single force equal to the vector sum of the forces – the superposition of forces

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots \vec{F}_n = \sum_i^n \vec{F}_i \tag{4.2}$$

The resulting force $\vec{F}$ is called *the net force*. The component version of the superposition principle can be written as

$$F_x = F_{x1} + F_{x2} + \cdots F_{xn} = \sum_i^n F_{xi} \tag{4.3}$$

$$F_y = F_{y1} + F_{y2} + \cdots F_{yn} = \sum_i^n F_{yi} \tag{4.4}$$

The magnitude and direction of the net force can be found from

$$F = \sqrt{F_x^2 + F_y^2}, \qquad \tan\theta = F_y/F_x \tag{4.5}$$

Forces act independently of each other: neither of them is modified by being applied at the same time as the other. Superposition works for any number of forces.

4.3 Newton's First Law

In the absence of external force, a particle moves with constant velocity $\vec{v}$

Here is an equivalent statement: In the absence of forces, a stationary particle remains stationary and a moving particle continues to move with unchanging speed in the same direction.

4.3.1 Inertial frames

Newton's formulation has two important implications:

1. Reference frames that move with constant velocities relative to each other are equivalent. These are called inertial frames of reference
2. Forces are the same in all inertial frames

State of rest from point-of-view of one observer is a state of constant velocity from point-of-view of another. It isn't mere motion that we need to explain – it is the change in state of motion (acceleration).

If non-zero net force is applied to a particle - all inertial observers see the same effect, viz. force is parallel to acceleration.

If acceleration is the same in all inertial frames, then force will be the same in all inertial frames

All forces behave in the same way: they all produce accelerations parallel to their directions.

4.4 Newton's Second Law

For any particle of mass m, the net force $\vec{F}$ on the particle is always equal to the mass m times the particle's acceleration:

$$\vec{F} = m\vec{a} \tag{4.6}$$

In this equation $\vec{F}$ denotes the vector sum of *all* forces on the particle and $\vec{a}$ is the particle's acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Second Law effectively defines mass: acceleration and force can be measured independently mass is then determined from Newton's second law.

Mass in Newton's second law is a measure of inertia. It is a scalar quantity called inertial mass. Mass is an inherent property of an object and is independent of the object's surroundings and of the method used to measure it.

Mass: The notation: m, the SI Unit: kg.

Force: The notation: $\vec{F}$, the SI unit: *newton*. We can use the standard kg to define the SI unit of force as

$$1\,N = 1\,kg\ \cdot m/s^2$$

Important:

1. Newton's second law is a vector equation.
   In practical applications it is convenient to use it in component form
   $$\sum F_x = ma_x$$
   $$\sum F_y = ma_y$$
2. The statement of Newton's second law refers to external forces, i.e. forces exerted on the body by other bodies in its environment
3. The mass m in $\vec{F} = m\vec{a}$ is constant
4. Newton's second law is valid only in inertial frames of reference.

Note: Newton formulated his second law in terms of momentum, i.e.

$$\vec{F} = \frac{d\vec{p}}{dt} \ \text{ where } \ \vec{p} = m\vec{v}$$

the rate of change of the momentum of a body is directly proportional to the net force acting on it, and the direction of the change in momentum takes place in the direction of the net force

4.4.1 Mass and Weight

There is a difference to be understood between the weight of an object and its inertia. (How hard it is to get it going is one thing, and how much it weighs is something else.) Weight and inertia are proportional, and on the earth's surface are often taken to be numerically equal, which causes certain confusion. On Mars, weights would be different but the amount of force needed to overcome inertia would be the same.

We use the term mass as a quantitative measure of inertia, and we may measure mass, for example, by swinging an object in a circle at a certain speed and measuring how much force we need to keep it in the circle. In this way we find a certain quantity of mass for every object.

Weight of a body is the force on it by the gravity $\vec{F}_g = \overrightarrow{W} = m\vec{g}$. Thus, the weight is a vector. Its SI unit is a newton (in British units the unit of weight is a *pound*).

When we buy in a grocery store - do we buy *mass* or *weight*?

Good to know: On Earth g depends on your altitude. On other planets, gravity will likely have an entirely new value, for example, on the moon 1.62 m/s^2, or 0.165*g.

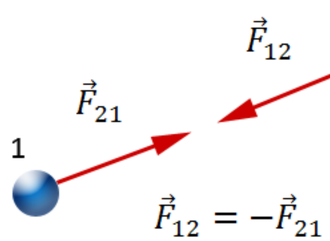
Attention: objects in orbit are not actually weightless; they do have weight since they do experience the force due to gravity, and are accelerated by it. Their state is correctly described as free-fall, when objects and everything in their environment is falling under the influence of gravity.

Attention: Heavier objects do not fall faster. It is the acceleration that defines how fast (not the force), namely $\vec{a} = \vec{F}_g/m = m\vec{g}/m = \vec{g}$ (the same acceleration for all objects in the absence of air resistance).

4.5 Newton's Third Law

A force of interaction on a body is always the result of interaction with another body, so forces always come in pairs.

> If object 1 exerts a force $\vec{F}_{21}$ on object 2, then object 2 always exerts force $\vec{F}_{12}$ on object 1 given by
>
> $$\vec{F}_{12} = -\vec{F}_{21}$$



or "*For every Action, there is an equal but opposite Reaction*".

Note that

1. Both *Action* and *Reaction* forces are the same physical origin
2. *Action* and *Reaction* act on different objects ($\vec{F}_{21}$ acts on 1, and $\vec{F}_{12}$ acts on 2)

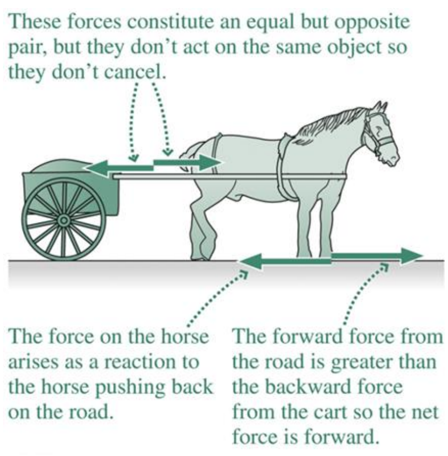
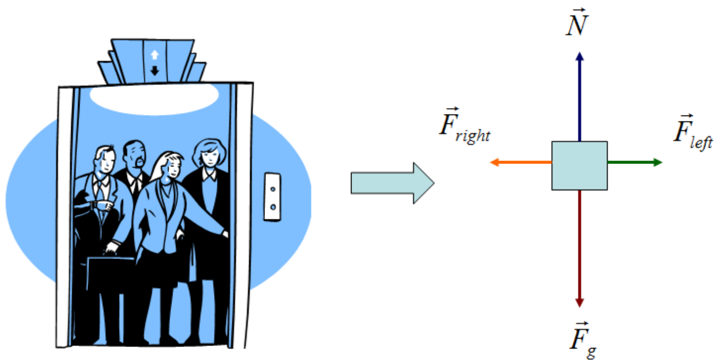


Figure 24 The horse-and-cart dilemma. The horse pulls on the cart, and the cart pulls back with a force of equal magnitude. So, how can the pair ever get moving? The net force on the horse involves forces from different third-laws pairs. Their magnitudes are not equal and the horse experiences a net force in the forward direction.

4.6 Free body diagrams

A free-body diagram is a graphical representation of all the forces acting on an object. Free-body diagrams are a powerful tool for solving problems. IDEA - replace an actual environment of an object as a set of forces acting on that object



Free-body diagrams show all forces acting on a particular body.

$$F = \vec{F}_1 + \vec{F}_2 + \cdots \vec{F}_n = \sum_{i}^{n} \vec{F}_i$$

Key ideas for drawing a free-body diagram:

1. Include ALL forces acting on the body matter.
2. When a problem includes more than one body – draw a separate free-body diagram for each body.
3. Not to include: any forces that the body exerts on any other body.
4. Not to include: non-existing forces (no object – no force).

When constructing a free-body diagram, it is a good idea to choose your coordinate system so that the motion of an object is along one of the axes.

Attention: So far we consider only translational motion (bodies cannot rotate), when all points of a body move with the same velocity (in the same direction with the same speed). Rotation of rigid bodies will be considered in chapters 9 – 11.

4.7 Examples

Example 4-1

An advertisement claims that a particular automobile can "stop on a dime". What net force would actually be necessary to stop a 1000-kg automobile (about 2,200 lb) travelling at 55 mph in a distance equal to the diameter of a dime, which is 1.6 cm?

SOLUTION:

1. Physics – Newton's second law and one dimensional motion with constant acceleration

2. The basic equations

$$\vec{F} = m\vec{a}$$

$$x = x_0 + v_0 t + \frac{at^2}{2}$$

$$v = v_0 + at$$

3. The motion is one dimensional, therefore we can use $F = ma$, and the final velocity of the car is $v = 0$.

To find the force we need to know the acceleration, which we can find from the kinematic equations

4. From the third equation $t = -v_0/a$, then the second equation is

$$x = x_0 - v_0\left(\frac{v_0}{a}\right) + \frac{1}{2}a\left(\frac{v_0}{a}\right)^2$$

$$2a(x - x_0) = -2{v_0}^2 + {v_0}^2$$

$${v_0}^2 = -2a(x - x_0)$$

The final velocity of the car is 0 m/s, then

$$a = -\frac{v_0^2}{2(x - x_0)}$$

The magnitude of the net force then is

$$F = m\frac{v_0^2}{2(x - x_0)}$$

5. Calculations

The initial data in SI units (we use 1 mile = 1609 m, 1 m = 100 cm, 1 h = 3600 s)

$$55\,mph = 55\,\frac{mile}{h}\left(\frac{1609\,m}{1\,mile}\right)\left(\frac{1\,h}{3600\,s}\right) = 24.59\,m/s$$

$$1.6\,cm = 1.6\,cm\left(\frac{1\,m}{100\,cm}\right) = 0.016\,m$$

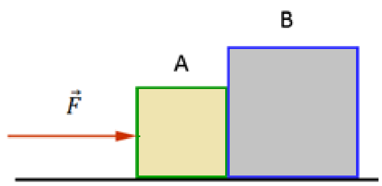
$$F = 1000\ kg \frac{(24.6\ m/s)^2}{2 * 0.016\ m} = 1.89 \cdot 10^7 N$$

6. Looking back.

The units are correct. The force is huge but on the other hand the stopping distance is very short (shorter than hitting a tree!). Therefore, the result looks reasonable but not the advertisement.

<u>Example 4-2</u>

Two boxes are lined up so that they are touching each other as shown in Figure. Box A has a mass of 20 kg, box B has a mass of 30 kg. An external force F=100 N pushes on box A toward the right.



a) find the acceleration of the boxes
b) find the force that box A exerts on box B

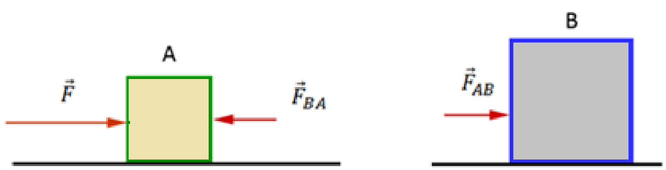
SOLUTION:

1. Physics – Newton's laws of motion for two objects

2. The basic equations

$F_{A,net} = m_A a \qquad F_{B,net} = m_B a$

3. Let's draw free-body diagrams (one for every block)



4. From the third law $\vec{F}_{BA} = -\vec{F}_{AB}$, then

$$F_{A,net} = F - F_{BA} = m_A a$$
$$F_{B,net} = F_{AB} = m_B a$$

Solving the system of equations for unknowns gives

$$a = \frac{F}{m_A + m_B}$$
$$F_{AB} = \frac{m_B}{m_A + m_B} F$$

5. Calculations

$$a = \frac{100\ N}{20\ kg + 30\ kg} = 2\ m/s^2$$

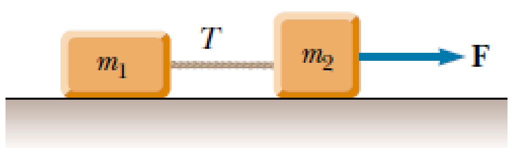
$$F_{AB} = \frac{30\ kg}{20\ kg + 30\ kg} 100\ N = 60\ N$$

6. Looking back

The units are correct. The magnitude of F_{AB} itself does not tell much, but we see that $F_{AB} < F$ as expected (because the external force $\vec{F}$ pushes two boxes and has to be large that the action-reaction force between the blocks.

Example 4-3

Two masses m_1 and m_2 situated on a frictionless, horizontal surface are connected by a light string. A force $\vec{F}$ is exerted on one of the masses to the right. Determine the acceleration of the system and the tension T in the string.



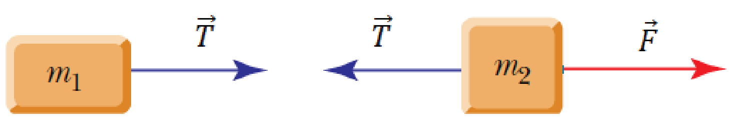
SOLUTION:

1. Physics – Newton's laws of motion for two objects

2. The basic equations

$F_{1,net} = m_1 a \qquad F_{2,net} = m_2 a$

3. Let's draw free-body diagrams (one for every block)



Then for the first block

$T = m_1 a$

and for the second block

$F - T = m_2 a$

4. Thus we get two equations with two unknowns T and a. The solutions are

$$a = \frac{F}{m_1 + m_2}, \qquad T = \frac{m_1}{m_1 + m_2} F$$

5. Calculations – no numbers, nothing to calculate

6. Looking back

The dimensions are correct. Let's also note that the solutions are remarkably similar to example 4-2.

<u>Example 4-4</u>

An aerostat of mass m starts coming down with a constant acceleration a. Determine the ballast mass to be dumped for the aerostat to reach the upward acceleration of the same magnitude. The air drag is to he neglected.

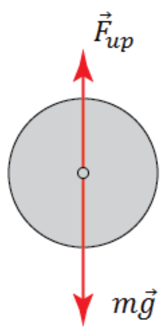
SOLUTION:

1. Physics – Newton's laws of motion for one object with two forces

2. The basic equations

$$F_{net} = ma$$

3. Let's draw free-body diagram



where $\vec{F}_g = m\vec{g}$ is the force of gravity and $\vec{F}_{up}$ is the lifting force. Then when the aerostat goes down

$F_{up} - mg = -ma$

and for goinf up we drop a mass m'

$F_{up} - (m - \mathrm{m}')g = (m - m')a$

4. Thus we have two equations with two unknowns, namely F_{up} and m'.

From the first equation $F_{up} = mg - ma$, then the second equation reads

$$mg - ma - (m - \mathrm{m}')g = (m - m')a.$$

Solving for m' gives

$$m' = 2m\frac{a}{g + a}$$

5. Calculations (no numbers – no calculations)

6. Looking back

The dimension of the answer [mass] is correct.

5 Applying Newton's Laws

5.1 Forces

A true understanding of Newton's laws requires a discussion of specific forces. In this chapter we deal with following forces

- Gravitational force $\vec{F}_g$
- Normal force $\vec{N}$
- Tension $\vec{T}$
- Frictional force $\vec{f}_\mu$
- Spring force $\vec{F}_s$

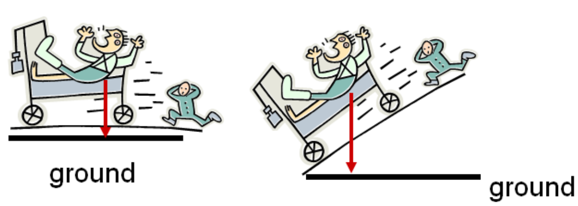
5.1.1 Gravitational force

A gravitational force on a body is a pull that is directed toward a second body. In this chapter we do not discuss the nature of this force, and we usually consider that the second body is Earth.

The magnitude:

$$F_g = mg \tag{5.1}$$

The direction is directly toward the center of Earth (toward the ground)



5.1.2 Normal force

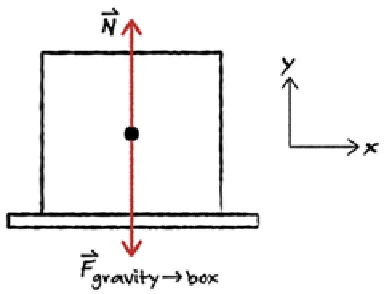
When a body presses against a surface, the surface deforms and pushes on the body with a normal force that is perpendicular to the surface. The normal force prevents the object from penetrating the surface.

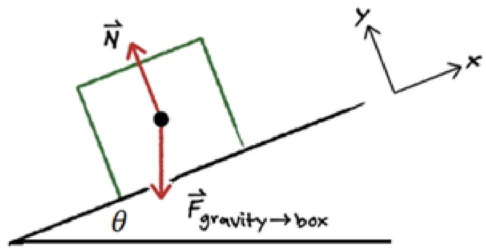
The magnitude: the component, perpendicular to the surface of contact, of the force exerted on an object (usually the force of gravity and any additional external force)

The direction: perpendicular and away from the surface

Example: A box on a flat surface A box on an incline

$\vec{N} = -\vec{F}_{gravity}$ $N = mg$ $N = F_{gravity}\cos\theta = mg\cos\theta$





Note: $mg\cos\theta$ is the component of the force of gravity perpendicular to the surface. It can be easily seen using geometry of similar triangles.

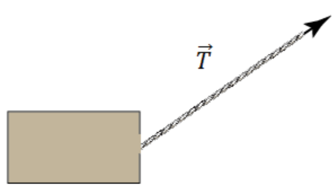
Attention: If there is external force acting on a body, then the magnitude of the normal force can be written as

$$N = mg\cos\theta \pm F_{external,\perp} \tag{5.2}$$

where $F_{external,\perp}$ is a perpendicular component of some external force (" $+$ " corresponds to a force pushing against the surface, and "$-$ " is for a force pulling the object from the surface).

5.1.3 Tension force

Tension is the pulling force exerted by a string, cable, chain, rope, or a similar object on a body. Normally the string is considered massless (comparing to the mass of body) and unstretchable (it is only a connection between bodies).



The symbol for the magnitude: T

The direction: away from the body and along the string

5.1.4 Frictional force

Microscopically, friction is a very complicated phenomenon.

Macroscopically, at large-scale level, it is relatively simple.

Phenomenologically (empirically), to a fairly good approximation, the frictional force is proportional to magnitude of normal force, and has a more or less constant coefficient

$$f = \mu N \tag{5.3}$$

where μ is called the coefficient of friction, and N is the normal force. Although this coefficient is not exactly constant, the formula is a good empirical rule for judging approximately the amount of force that will be needed in certain practical or engineering circumstances. If the normal force or the speed of motion gets too big, the law fails because of the excessive heat generated. It is important to realize that each of these empirical laws has its limitations, beyond which it does not really work.

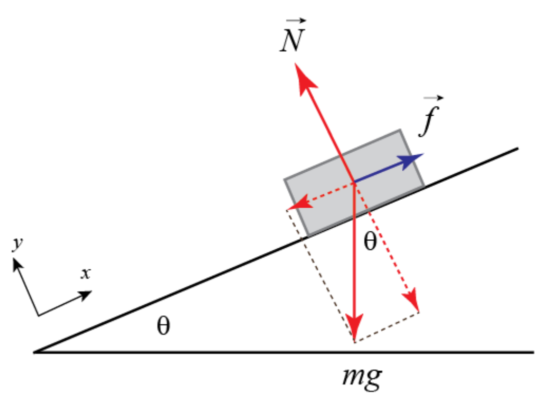
Static friction (a body does not move):

1. If the body does not move, then the static frictional force $\vec{f_s}$ and the component of $\vec{F}$ that is parallel to the surface balance each other.
2. *The magnitude* of $\vec{f_s}$ has a maximum value that is given by $f_{s,max} = \mu_s N$

where μ_s is a coefficient of static friction, and N is the magnitude of the normal force on the body from the surface.

Direction: parallel to the surface, and is directed opposite the component of an external force.

One must overcome (exceed) the force of static friction it in order to initiate motion of the body along the surface.



Here is a simple way to determine the coefficient of static friction. Let's consider an object on an incline.

When block is not moving, friction force compensates x −component of gravitational force:

$$f_s = mg \sin\theta$$

However by definition $f_s = \mu_s N = \mu_s mg \cos\theta$ then $mg \sin\theta = \mu_s mg \cos\theta$. Let's θ_{max} is the largest angle when the block is still not moving, then

$$\mu_s = \tan\theta_{max} \tag{5.4}$$

Thus, equation (5.4) provides a way to measure a value of static frictional coefficient. Note that μ_s can be greater than 1.

Kinetic friction (a sliding body)

If the body begins to slide along the surface, the magnitude of the frictional force rapidly decreases to a value f_k given by

$$f_k = \mu_k N$$

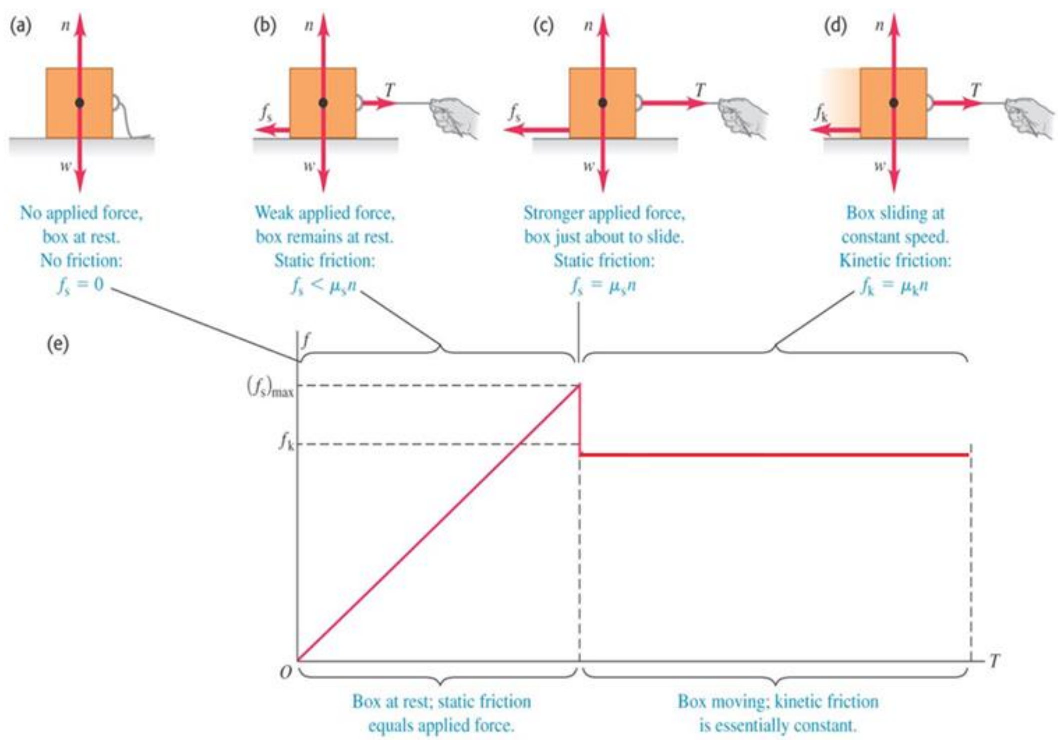
where μ_k is the coefficient of kinetic friction. Note that the kinetic friction does not depend on speed.

Direction: always opposite to the direction of velocity

Useful note: *Frictional force is independent of the area of contact between the body and the surface.*

Approximate coefficients of static and kinetic friction can be found in many textbooks as well as on the Web.

Overall:



Rolling Friction[1]

It's a lot easier to move a loaded filing cabinet across a horizontal floor using a cart with wheels than to slide it. How much easier? We can define a coefficient of rolling friction which is the horizontal force needed for constant speed on a flat surface divided by the upward normal force exerted by the surface, or $f_r = \mu_r N$. Transportation engineers call the tractive resistance. Typical values of are 0.002 to 0.003 for steel wheels on steel rails and 0.01 to 0.02 for rubber tires on concrete. These values show one reason railroad trains are generally much more fuel efficient than highway trucks.

[1] from University Physics by Young and Freedman

5.1.5 Fluid resistance and terminal speed

A fluid is anything that can flow - generally either a gas or a liquid. A body moving through a fluid exerts a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the body with an equal and opposite force.

The direction of the fluid resistance force acting on a body is always opposite the direction of the body's velocity relative to the fluid. The magnitude of the fluid resistance force usually increases with the speed of the body through the fluid. This is very different from the kinetic friction force between two surfaces in contact, which we can usually regard as independent of speed.

For *very low speeds*, the magnitude f of the fluid resistance force is approximately proportional to the body's speed v:

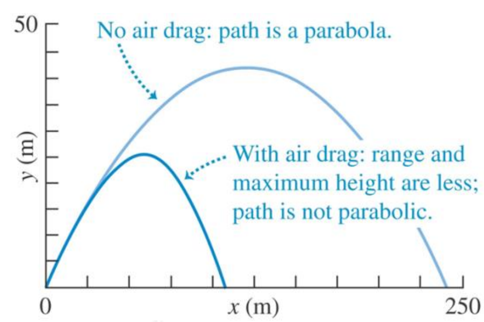
$$f = bv$$

where b is a proportionality constant that depends on the shape and size of the body and the properties of the fluid.

In motion through air *at high speeds* (e.g. the speed of a tossed tennis ball or faster), the resisting force is approximately proportional to v^2 rather than to v. It is then called **air drag** or simply drag. The magnitude of the drag force is related to the relative speed by

$$f = \frac{1}{2} C\rho A v^2$$

where C is experimentally determined drag coefficient, ρ is the air density (mass per volume) and A is the effective cross-sectional area of the body (the area of a cross section taken perpendicular to the velocity $\vec{v}$). The drag coefficient C (typical values range from 0.4 to 1.0) is not truly a constant for a given body, because if v varies significantly, the value of C can vary as well. Here, we ignore such complications.



When an object falls from rest through air, the drag force f is directed upward; its magnitude gradually increases from zero as the speed of the body increases. This upward force opposes the downward gravitational force on the body.

$$f - mg = -ma$$

where m is the mass of the body. If the body falls enough, f eventually equals mg, this means that $a = 0$ and so the body's speed no longer increases. The body then falls at a constant speed, called the terminal speed v_t. From

$$\frac{1}{2}C\rho A v^2 = mg$$

follows

$$v_t = \sqrt{\frac{2mg}{C\rho A}}$$

This expression for terminal speed explains why heavy objects in air tend to fall faster than light objects. Two objects with the same physical size but different mass (say, a table-tennis ball and a lead ball with the same radius) have the same value of A but different values of m. The more massive object has a higher terminal speed and falls faster. The same idea explains why a sheet of paper falls faster if you first crumple it into a ball; the mass is the same, but the smaller size makes A smaller thus smaller drag force f (less air drag for a given speed) and v_t larger.

Table
Terminal Velocity

Object	m/s
Shot put	150
Skydiver	60
Hailstone	15
Ping-Pong ball	9
Raindrop	7
Parachutist	5

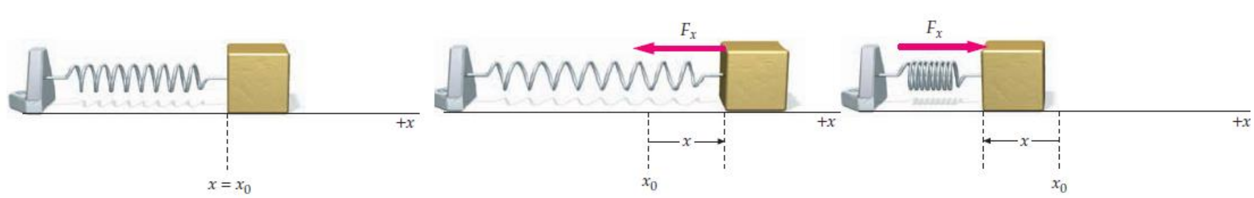
Note: normally in introductory physics classes we do not deal with the drag force. However, knowing about it is vital for understanding effects of air resistance in everyday life.

5.1.6 Spring Force

When a spring is stretched from its unstressed length by a distance x, the force it exerts is found experimentally to be

$$F_s = -k(x - x_0) \tag{5.5}$$

where the positive constant k, called the spring constant, is a measure of the stiffness of the spring, and x_0 is the equilibrium position (normally counted as $x_0 = 0$). A negative value of x means the spring has been compressed a distance $|x|$ from its unstressed length. The negative sign means that when the spring is stretched (or compressed) in one direction, the force it exerts is in the opposite direction. This relation is known as Hooke's law. For small displacements, nearly all restoring forces obey Hooke's law.



5.2 Dynamics of circular motion

In chapter 3 we talked about uniform circular motion when a body moves in a circle (or a circular arc) at constant *speed* v. In this case the body has a centripetal acceleration (directed toward the center of the circle) of constant magnitude

$$a = \frac{v^2}{R} \tag{5.6}$$

where R is the radius of the circle.

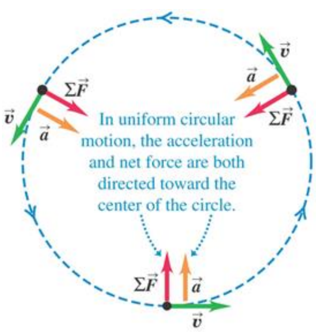
A centripetal force accelerates a body by changing the direction of the body's velocity without changing the body's speed.

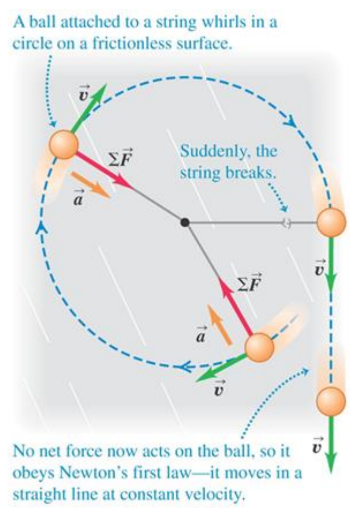
The magnitude of a centripetal force (or a net centripetal force) is

$$F_{net} = m\frac{v^2}{R} \tag{5.7}$$

Because the speed v is constant, so are also the magnitudes of the acceleration and the force.

However, *the direction of the centripetal acceleration and force* are not constant; they vary continuously so as to always to point toward the center of the circle.



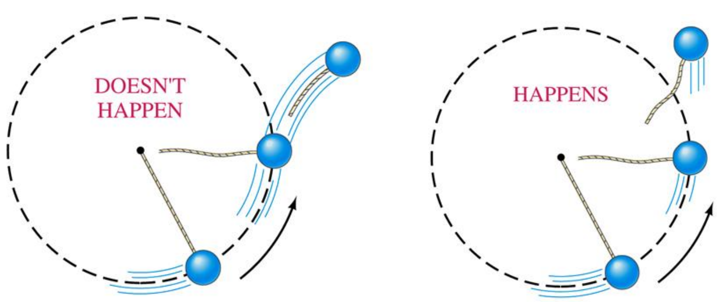


Tension in the string causes the ball to move in circular path with constant speed.

Important: tension (force) pulls the ball toward the center, rather than prevent it from moving in a straight line.

If the string breaks, the ball would move along a straight line tangent to circular path.

Important: when released (see the difference in trajectories)



5.3 Few guidelines for solving most common problems in “Applying Newton’s Laws”

As always, it is very productive to follow this procedure for problem solving.

1. Name the type of physics most likely related to the problem in hand. Draw a diagram if needed.
2. Write down the "basic" equations for the physics of the problem.
3. Simplify (when possible) the "basic" equations using given data and conditions.
4. Solve the "adjusted" equation for the unknown(s) using algebra, trigonometry and calculus.
5. Achieve a numerical answer using your symbolic solution and the proper units.
6. Step back, and evaluate your answer in terms of units, dimensions, and most importantly, common sense.

Since this subject offers so many variations of problems, let elaborate more on most important steps, namely steps 2 and 3 in the above list.

Many problems on “Applying Newton’s Laws” are either “pure force problems” or a combination “a force problem” + “a motion with constant acceleration problem”. In the following we are going to concentrate “force problems”. Here are most important points that may help you to develop a systematic approach to attacking problems from simple ones to more challenging.

Point 1: Ask yourself a question - How many objects do I need to consider? Normally, in physics 231 we deal with one or two objects (having three or four objects are much less common problems).

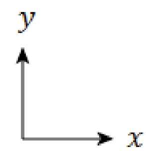
Point 2: Draw free-body diagrams. One diagram for every object! You may find it helpful to check every force from the table below. And if you do have it, then put it on your free-body diagram with a proper direction.

Summary of forces

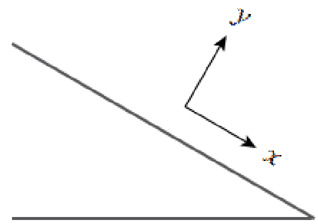
Force	Magnitude	Direction
Gravity	$F_g = mg$	toward the ground
Normal	$N = mg\cos\theta \pm F_{external,\perp}$	perpendicular to the surface (and away)
Tension	T	along the string
Friction (static)	$f_{s,max} \leq \mu_s N$	opposite the component of external force and parallel to the surface
Friction (kinetic)	$f_k = \mu_k N$	always opposite to the direction of velocity
Centripetal force	$F_c = m\dfrac{v^2}{r}$	toward the center of rotation
Spring (or elastic)	$F_s = -k(x - x_0)$	opposite to stretch/compression direction

Point 3: Choose proper coordinate system for every object. Normally we work with the Cartesian coordinate system. Choosing wisely helps to reduce algebra and trigonometry load.

For most problems we use the regular orientation (horizontal direction for x and vertical one for y).



However, for incline problems we choose a 'rotated' coordinate system



Sure, you may solve incline problems with the regular orientation as well, but then you have much more trigonometry on your hands.

Point 4: Once we identify all forces and their directions, we write Newton's second law $\vec{F}_{net} = m\vec{a}$ for every $x-$ and $y-$component of every object.

$$F_{net,x} = \sum F_x = ma_x$$
$$F_{net,y} = \sum F_y = ma_y$$

For objects in equilibrium (that are stationary objects or objects moving with constant velocity) we apply $\vec{a} = 0$, or $\vec{F}_{net} = 0$. Most problems in Physics I deal with either *linear* motion or *rotational* motion. For uniform rotational motion we write Newton's second law as

$$F_{net} = m\frac{v^2}{r}$$

More explicitly, for linear problems (objects moving along a straight line) we have

4a: One object:

$$F_{net,x} = ma_x$$
$$F_{net,y} = ma_y$$

4b: Two connected objects (one set of equations for every object).

$$F_{net1,x} = ma_{1x} \qquad\qquad F_{net2,x} = ma_{2x}$$
$$F_{net1,y} = ma_{1y} \qquad\qquad F_{net2,y} = ma_{2y}$$

Attention: the objects share the same tension (if they are connected by a massless cable) and the same acceleration in proper directions, for example a magnitude of an acceleration for the first object along x can be equal to acceleration for the second object along y, or $a_{1,x} = a_{2,y}$. Also, remember Newton's third law or action-reaction pairs for interacting objects and watch for directions of motion.

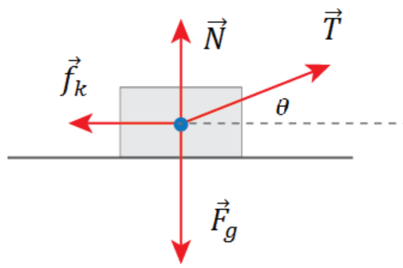
4c: For uniform rotational motion (one object only)

horizontal plane or vertical plane

$$F_{net,x} = \pm\frac{mv^2}{r} \qquad F_{net,y} = \pm\frac{mv^2}{r}$$

Here are a couple cases representing many common problem.

Case 5-1: an object on a surface pulling along by force of tension

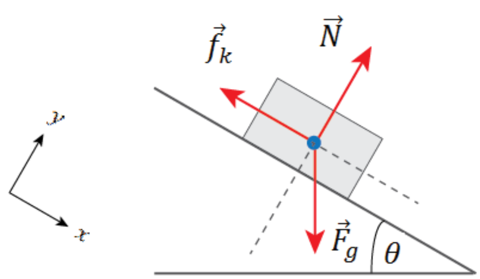


We have one object with four forces acting upon it, and we use the regular coordinate system, then

$$F_{net,x} = T\cos\theta - f_k = ma$$
$$F_{net,y} = N + T\sin\theta - mg = 0$$

with $f_k = \mu_k N$. Then we solve the system for unknowns (pure algebra). Note that, if the object is not moving then $a = 0$, and instead of f_k we use f_s.

Case 5-2: an object on an incline moving *downhill*

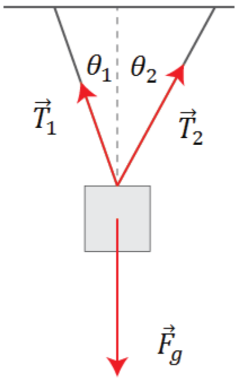


One object, three forces, a rotated coordinate system

$$F_{net,x} = mg\sin\theta - f_k = ma$$
$$F_{net,y} = N - mg\cos\theta = 0$$

with $f_k = \mu_k N$

Case 5-3: an object attached to two cables (one cable case is way too simple)



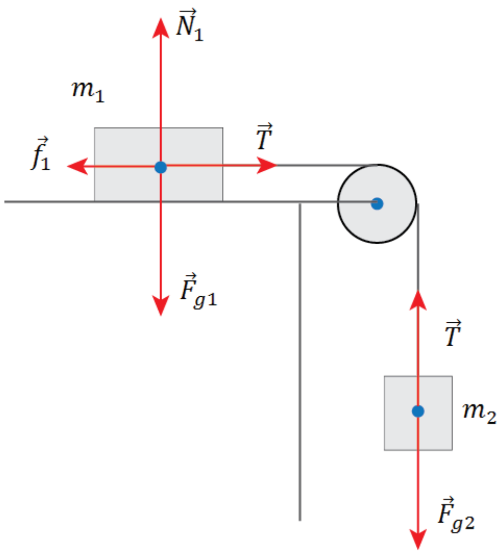
There is one object + three forces (gravity and two tensions).

The equations

$$F_{net,x} = T_2 \sin\theta_2 - T_1 \sin\theta_1 = 0$$
$$F_{net,y} = T_1 \cos\theta_1 + T_2 \cos\theta_2 - mg = 0$$

Then we solve for unknowns (normally T_1 and T_2). Note that the angles can be counted either from vertical (like in the figure) or from horizontal.

Case 5-4: two connected objects (and moving)



Two objects, many forces (four on the first object and two on the second), regular coordinate system

For the first object

$$F_{net1,x} = T - f_1 = m_1 a$$
$$F_{net1,y} = N_1 - m_1 g = 0$$

For the second object

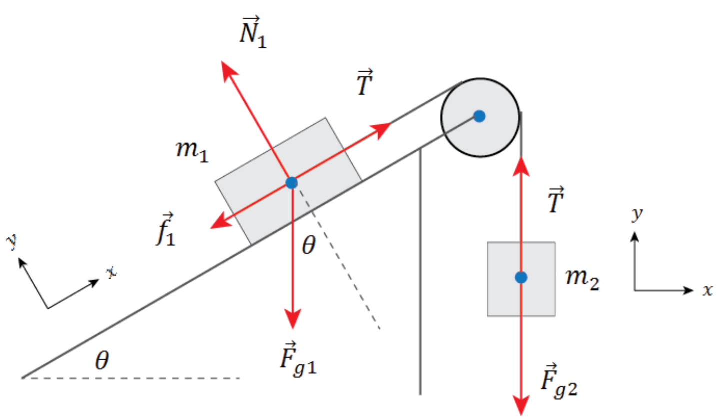
$$F_{net2,x} = 0$$
$$F_{net2,y} = T - m_2 g = -m_2 a$$

The definition for the frictional force provides one more equation

$f_1 = \mu_1 N_1$

Then we solve the system for unknowns.

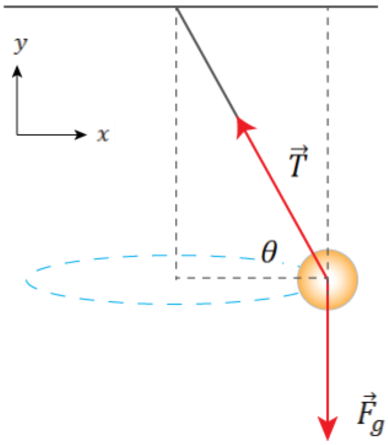
If the first body was on incline like on the figure below, then we keep everything the same for the second object, but we need to do the following changes for the first one. First we choose a rotated coordinates for the first body. Then the equations are



$$F_{net1,x} = T - f_1 - m_1 g \sin\theta = m_1 a$$
$$F_{net1,y} = N_1 - m_1 g \cos\theta = 0$$

with $f_1 = \mu_1 N_1$

Case 5-5: A ball on a rope that is in circular uniform motion in a horizontal plane, with the rope making an angle to the horizontal.



One object, two forces, uniform circular motion

Equations

$$-T\cos\theta = -\frac{mv^2}{r} \qquad along\ x$$

$$T\sin\theta - mg = 0 \qquad along\ y$$

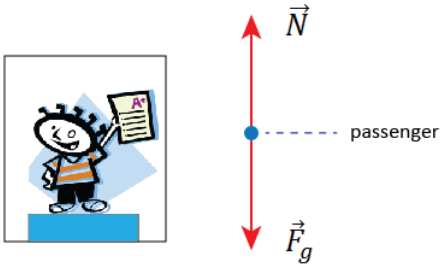
5.4 Examples

Example 5-1

A passenger of mass m stands on a platform scale in an elevator cab.

c) Find a general solution for the scale reading, whatever the vertical motion of the cab

d) What does the scale read is the cab is moving upward at a constant 1.5 m/s² and m=80 kg.



1. Physics – 1D (vertical motion), newton's laws of motion

2. Newton's second law

$$\vec{F} = m\vec{a}$$

3. Scales measure weight by reading N (weight is directed down but the normal force is up and perpendicular to the surface of the scale)

The free-body diagram can be seen on the left

Equation of motion (in y −direction $F_{net,y} = ma_y$)

$$N - mg = \pm ma$$

for any choice of acceleration (a is positive for upward acceleration and negative for downward acceleration).

4. The equation can be easily solved as

$$N = m(g \pm a)$$

5. Calculations for upward motion

$$N = 80\ kg \cdot (9.8 + 1.5)\ m/s^2 = 904\ N$$

Example 5-2

A curious student dangles her cell phone from a thin piece of string while the jetliner she is in takes off. She notices that the string makes and angle of 25^0 with respect to the vertical while the aircraft accelerates for takeoff, which takes about 18 seconds. Estimate the takeoff speed of the aircraft

SOLUTION:

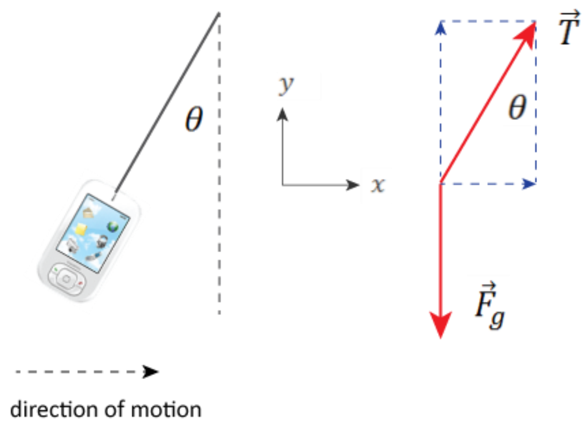
1. Physics – one dimensional motion with constant acceleration and Newton's second law.

2. For one 1D motion with constant acceleration

$$x = x_0 + v_0 t + \frac{at^2}{2} \qquad v = v_0 + at$$

and Newton's second law

$$\vec{F} = m\vec{a}$$



3. If we knew the acceleration we could estimate the speed from using the kinematic equations.

Free-body diagram for the cell phone

The system of equations

$$\sum F_x = T\sin\theta = ma$$
$$\sum F_y = T\cos\theta - mg = 0$$

4. The linear system of equations has two unknowns T and a. From the second equation $T = mg/\cos\theta$, then

$$a = \frac{T\sin\theta}{m} = \frac{mg\sin\theta}{m\cos\theta} = g\tan\theta$$

Using the second kinematic equation we have

$$v = at = g\tan\theta \cdot t$$

5. Calculations

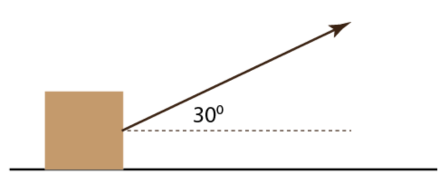
$$v = 9.8\ m/s^2 \cdot \tan 25^0 \cdot 18\ s = 82\ m/s$$

6. We have right units for the speed.

The takeoff speed of 82 m/s is about 295 km/h (or 183 mph). It sounds as a realistic speed for a medium size airplane.

Example 5-3

Suppose you try to move a crate pulling upward on the rope at an angle 30^0 above the horizontal. How hard do you have to pull to keep the crate moving with constant velocity? Is it easier or harder than puling horizontally? Assume: the crate is 50 kg and the coefficient of kinetic friction is 0.4.



SOLUTION:

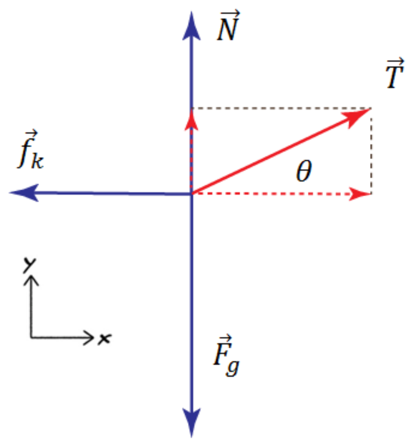
1. Physics – Newton's laws

2. Newton's second law (in component form) for motion with constant velocity

$$\sum F_x = 0$$
$$\sum F_y = 0$$

3. There is one object with four forces acting upon it: gravity, normal, tension, friction

Let's draw a free-body diagram for the crate



The system of equations

$$\sum F_x = T\cos\theta - f_k = 0$$
$$\sum F_y = N + T\sin\theta - mg = 0$$

the kinetic frictional force is defined as

$$f_k = \mu_k N$$

4. The linear system with two equations has two unknowns T and N.

The system can easily be solved using substitution. From 2nd equation

$$N = mg - T\sin\theta$$

then $f_k = \mu_k(mg - T\sin\theta)$ and from equation for the x-component

$$T\cos\theta - \mu_k mg + \mu_k T\sin\theta = 0$$
$$T(\cos\theta + \mu_k\sin\theta) = \mu_k mg$$
$$T = \frac{\mu_k mg}{\cos\theta + \mu_k\sin\theta}$$

5. Calculations

$$T = \frac{0.4 \cdot 50kg \cdot 9.8\,m/s^2}{\cos 30^0 + 0.4 \cdot \sin 30^0} = 184\,N$$

6. The units are correct (newton for the force).
If the angle was 0^0 then the needed tension would be 196 N. Thus we need less force pulling at some angle. It is interesting to find what pulling angle would need less force. It happens when $\cos\theta + \mu_k \sin\theta$ takes the largest value

$$\frac{d}{d\theta}(\cos\theta + \mu_k \sin\theta) = \mu_k \cos\theta - \sin\theta = 0$$

and $\tan\theta = \mu_k$. For $\mu_k = 0.4$ the angle is about 22^0, and the smallest force is 182 N.

Example 5-4

A runaway truck with failed brakes is moving 108 km/h just before the driver steers the truck up the runaway ramp with an inclination of 30^0. Assume that the coefficient of friction between the ramp and the truck is 0.6.

a) What minimum length L must the ramp have to stop along it?
b) How long (in seconds) does it take for the truck to stop?
c) Does the minimum length L increase, decrease, or remain the same for a small passenger car?

SOLUTION

1. Physics – 1D motions with constant acceleration along the ramp, Newton's second law

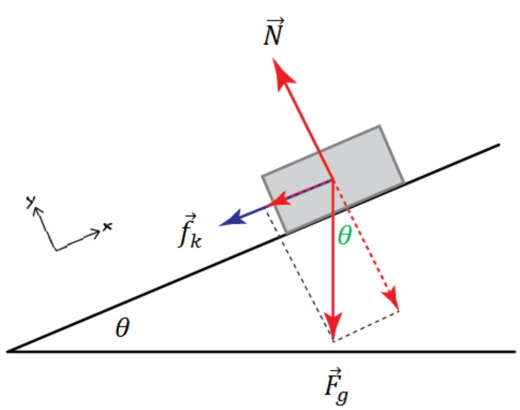
2. For one 1D motion with constant acceleration

$$x = x_0 + v_0 t + \frac{at^2}{2} \qquad v = v_0 + at$$

and Newton's second law

$$\vec{F} = m\vec{a}$$

3. First we need to find the deceleration of the truck on the ramp. Then we can solve the kinematic problem to find the stopping distance and time.



The free-body diagram includes all three forces acting on the truck (gravity, normal, and friction). We choose the coordinate system as a rotated one with the x-coordinate parallel to the ramp.

The system of equations

$$\sum F_x = -mg\sin\theta - f_k = ma$$
$$\sum F_y = N - mg\cos\theta = 0$$

The kinetic frictional force is defined as

$$f_k = \mu_k N$$

4. The linear system of equations has two unknowns (acceleration and normal force). Solving the system of equation for the acceleration gives (note: from simple geometry follows that the black θ is equal to the green θ angle)

$$N = mg\cos\theta$$
$$-mg\sin\theta - \mu_k mg\cos\theta = ma$$
$$a = -g(\sin\theta + \mu_k\cos\theta)$$

Now we can work with the kinematic equations. Note that a is the deceleration and the kinetic equations

$$x = x_0 + v_0 t + \frac{at^2}{2} \qquad v = v_0 - at$$

Since the final velocity of the truck is zero at the end of the ramp, then we can easily find the stopping time and then the distance

$$t = -\frac{v_0}{a}, \;\; x = -v_0\frac{v_0}{a} + \frac{a}{2}\frac{v_0^2}{a^2} = -\frac{v_0^2}{2a}$$
$$x = \frac{v_0^2}{2g(\sin\theta + \mu_k\cos\theta)}$$

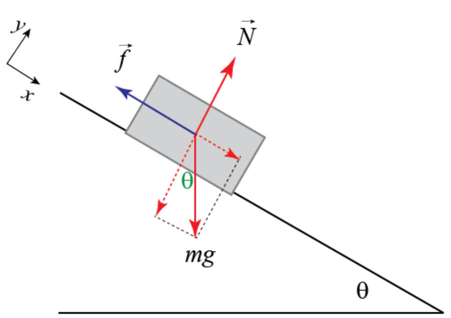
5. Calculations

In SI units

$$108\ km/h = 108\,\frac{km}{h}\left(\frac{1000\,m}{1\,km}\right)\left(\frac{1\,h}{3600\,s}\right) = 30\,m/s$$
$$a = 9.8\,m/s^2\cdot(\sin 30^0 + 0.6\cdot\cos 30^0) = 10\,m/s^2$$
$$t = \frac{30\,m/s}{10\,m/s^2} = 3\,s \qquad x = \frac{(30\,m/s)^2}{2\cdot 10\,m/s^2} = 45\,m$$

6. Proper units for the time and distance. Both the time and distance look realistic.

What if the truck was going downhill? How does friction affect the stopping distance? In this case the direction of the frictional force is in the opposite and then



$$\sum F_x = mg\sin\theta - f_k = ma$$
$$\sum F_y = N - mg\cos\theta = 0$$

$$mg\sin\theta - \mu_k mg\cos\theta = ma$$
$$a = g(\sin\theta - \mu_k\cos\theta)$$

finally

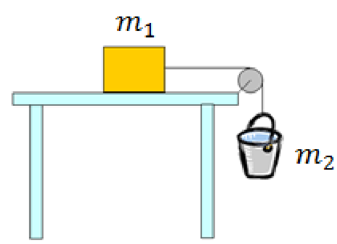
$$x = \frac{v_0^2}{2g(\mu_k\cos\theta - \sin\theta)}$$

The truck cannot stop if $\mu_k\cos\theta - \sin\theta \le 0$.

Example 5-5

A 12.0-kg box (m_1) is connected to an empty 2.00-kg bucket (m_2) by a cord running over a very light frictionless pulley. There is no appreciable friction on the box, since somebody spilled oil on the table. The box starts from rest.

a) Find the acceleration of the box and the bucket.
b) What is the tension in the cord?



SOLUTION:

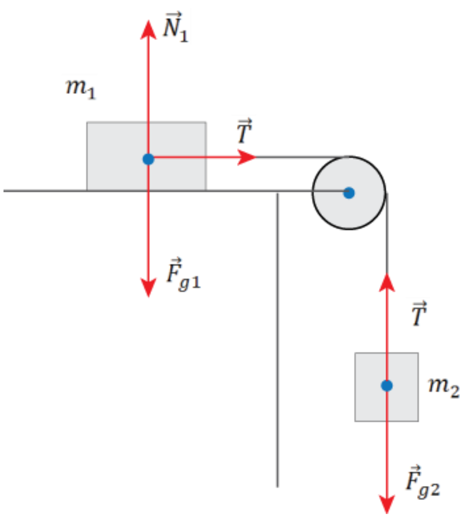
1. Physics: Newton's laws for two connected objects.

2. Second Newton's law for 2D case

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y$$

3. We have two connected objects; therefore we have to write the second law for every object.

Let's start with free-body diagrams



The system of equations for two objects

$$\begin{array}{ll} T = m_1 a & m_1 \, along \; x \\ N_1 - m_1 g = 0 & m_1 \; along \; y \\ T - m_2 g = -m_2 a & m_2 \; along \; y \end{array}$$

4. Note that according to Newton's third law the tension in the cable in the same for both objects (if the cable does not have mass). The linear system of three equations has three unknowns. Solving by substitution gives

$$a = \frac{m_2}{m_2 + m_1} g \qquad T = \frac{m_2 m_1}{m_2 + m_1} g$$

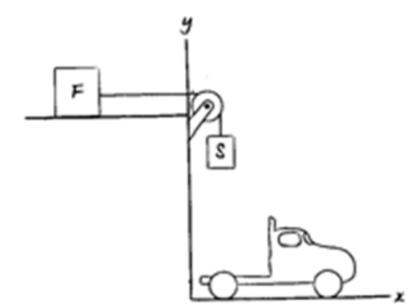
5. Calculations

$$a = \frac{2\;kg}{2\;kg + 12\;kg} 9.8\;m/s^2 = 1.4\;m/s^2\,, \qquad T = 12\;kg \cdot 1.4\;m/s^2 = \;16.8\;N$$

6. Looking back

We have proper units for the acceleration and tension. The numbers do not tell much if we can trust the results, but we may consider an extreme case when $m_1 = 0$, then $a = 9.8\;m/s^2 \quad T = 0\;N$. Correct. If $m_2 = 0$, then $a = 0$ also correct.

Example 5-6



Bank robbers have pushed a 1000 kg safe to a third-story floor-to-ceiling window. They plan to break the window and lower the safe 10.0 m to their truck. Not being too clever, they stack up 500 kg of furniture, tie a rope between the safe and the furniture, and place the rope over a pulley. Then they push the safe out of the window. The coefficient of kinetic friction between the furniture and the floor is 0.5. The rope would break if the force on it exceeds 5000 N.

a) Does the rope break?
b) What is the acceleration of the safe?
c) What is the safe's speed when it hits the truck?

SOLUTION:

1. Physics: Newton's laws for two connected objects. This problem is practically identical to problem in example 5-5 but with friction.

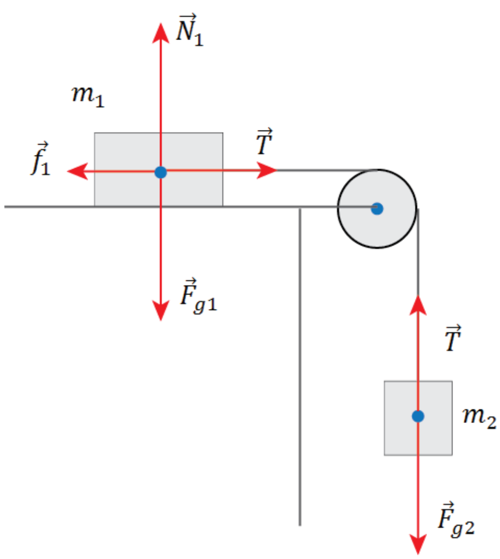
2. Second Newton's law for 2D case for every object

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y$$

and 1D motion with constant acceleration

$$v^2 = {v_0}^2 + 2a(y - y_0)$$

3. We have two connected objects; therefore we have to write the second law for every object. Let's start with free-body diagrams



For the first object (the furniture)

$$F_{net1,x} = T - f_1 = m_1 a$$
$$F_{net1,y} = N_1 - m_1 g = 0$$

For the second object (the safe)

$$F_{net2,x} = 0$$
$$F_{net2,y} = T - m_2 g = -m_2 a$$

The definition for the frictional force provides one more equation

$$f_1 = \mu_1 N_1$$

4. Note that according to Newton's third law the tension in the cable in the same for both objects (if the cable does not have mass). From $F_{net1,y} = N_1 - m_1 g = 0$ we have $N_1 = m_1 g$, then $f_1 = \mu_1 N_1 = \mu_1 m_1 g$. And we have a system of two equations with two unknowns

$$T - \mu_1 m_1 g = m_1 a$$

$$T - m_2 g = -m_2 a$$

Solving by substitution gives

$$a = \frac{m_2 - \mu_1 m_1}{m_2 + m_1} g \qquad T = \frac{m_2 m_1}{m_2 + m_1}(1 + \mu_1) g$$

The speed can be found from

$$v^2 = {v_0}^2 + 2a(y - y_0)$$

where we use a from equation above if the rope does not break, or $a = g = 9.8\, m/s^2$ if the rope breaks.

5. Calculations

$$T = 4900\, N, \quad a = 4.9\, m/s^2$$

Thus, the rope does NOT break. The safe's speed at the end is

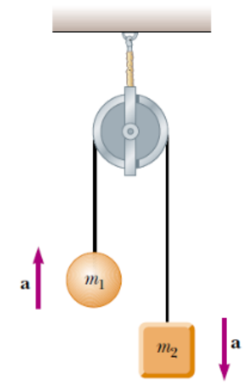
$$v = 9.9\, m/s^2$$

6. Looking back

We have proper units for the acceleration and tension. In the absence of friction ($\mu_1 = 0$) our equations for the acceleration and tensions are identical to equations from example 5-5.

Example 5-7

Two masses m_1 and m_2 are connected by a light string that passes over a frictionless pulley, as in Figure.



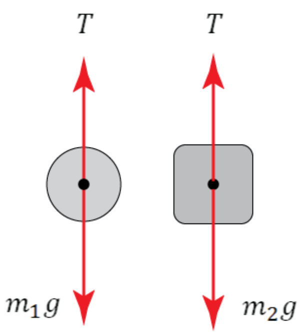
Determine

a) the tension in the string,
b) the acceleration of each object,
c) the force that the ceiling exerts on the hook

SOLUTION:

1. Physics: Newton's laws for two connected objects. (Atwood machine)

2. Second Newton's law for 2D case for every object



3. We have two connected objects; therefore we have to write the second law for every object.

Let's start with free-body diagrams.

This is a system of equations for two objects. Note, for every object we have only one coordinate, namely y- component

$$T - m_1 g = m_1 a$$

$$T - m_2 g = -m_2 a$$

4. Note that according to Newton's third law the tension in the cable in the same for both objects (if the cable does not have mass).

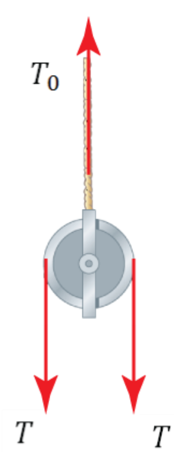
Attention, we should be careful with signs for the acceleration in both equations. We do not know what mass is large, and in what direction the masses will move. Thus we assume that the first mass moves upward (positive acceleration), and the second mass moves downward (negative). Actually it is not important; a proper sign for acceleration a will be derived automatically by solving the system above.

From the first equation $T = m_1 g + m_1 a$, substituting it into the second equation gives $m_1 g + m_1 a - m_2 g = -m_2 a$, then $(m_1 + m_2)a = (m_2 - m_1)g$

$$a = \frac{m_2 - m_1}{m_2 + m_1} g$$

and the tension (after simple algebra) is

$$T = \frac{2 m_1 m_2}{m_1 + m_2} g$$



The force on the hook can be found from the free-body diagram on the right.

$$T_0 - T - T = 0, \quad T_0 = 2T = \frac{4 m_1 m_2}{m_1 + m_2} g$$

6. Looking back

Let's see what we get if $m_1 = m_2 = m$. In this case we get $a = 0$, $T = mg$ and $T_0 = 2mg$ as we would expect for this balanced case.

For $m_2 \gg m_1$ he have $a \approx g$, (a freely falling body) and $T \approx 2m_1 g$.

<u>Example 5-8</u>

A 50-kg refrigerator is placed on the flat floor of a truck. The coefficients of friction between the refrigerator and floor are μ_s=0.24 and μ_k=0.21. The truck starts to move with an acceleration of 2.5 m/s^2. If the refrigerator is 2.0 m from the rear of the truck when the truck starts, how much time elapses before the refrigerator falls off the truck? How far does the truck travel in this time?

SOLUTION

1. Physics – Newton's laws, 1D motion with constant acceleration

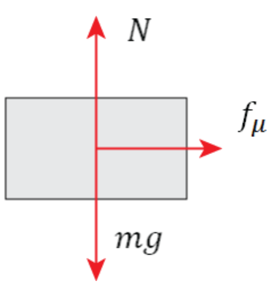
2. Equations

$\sum F_x = ma_x$, $\quad f_s = \mu_s N$, $\quad f_k = \mu_k N$

$$x = x_0 + v_0 t + \frac{at^2}{2}$$

$$v = v_0 + at$$

3. First we work with the forces acting on the refrigerator.



Assume that the truck moves to the right. Then it is the frictional force that provides acceleration for the refrigerator! The most acceleration we can get is $a_s = f_s/m$. Since $N = mg$, then $a_s = \mu_s g$. However, as soon as the truck starts moving with acceleration larger than a_s, the refrigerator starts sliding on the truck's bed. Now we deal with kinetic friction, that is equal to $a_k = \mu_k g$. Now, the truck moves with a acceleration but the refrigerator has $a_k < a$ acceleration, or the truck moves faster than the refrigerator with acceleration $a_{rel} = a - a_k = a - \mu_k g$. Having this acceleration we can find how much time it will take for the refrigerator to move along the truck's bed. From 1D motion with constant acceleration

$$x = x_0 + v_0 t + \frac{at^2}{2}, \quad v_0 = 0, \qquad x - x_0 = x_{bed} = \frac{a_{rel} t^2}{2}, \quad t = \sqrt{2\frac{x_{bed}}{a_{rel}}} = \sqrt{2\frac{x_{bed}}{a - \mu_k g}}$$

During this time the truck will move

$$x_{truck} = \frac{at^2}{2} = a\frac{1}{2}2\frac{x_{bed}}{a - \mu_k g} = \frac{a}{a - \mu_k g} x_{bed}$$

5. Calculations

$$t = 3.0\ s, \quad x_{truck} = 11\ m$$

6. Looking back.

The dimensions are correct and the numbers looks sensible.

Note that this problem has very little work with equations, but careful thinking is needed.

<u>Example 5-9</u>

In a loop-the-loop stunt a stuntman is riding a bicycle. Assuming that the loop is the circle with radius R=2.7 m, what is the least speed the stuntman has to have at the top of the loop to remain in contact with it there.

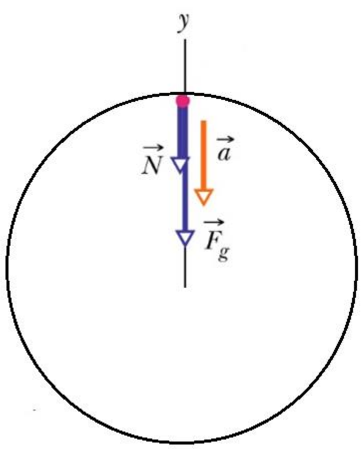
SOLUTION

1. Physics – circular motion and Newton's laws

2. Equation of motion

$$F_{net} = ma = \frac{mv^2}{R}$$

3. A free-body diagram with all forces



Then

$$-N - mg = m(-a)$$

4. This becomes

$$-N - mg = m\left(-\frac{v^2}{R}\right)$$

If the stuntman has the least speed to remain in contact, then he is on the verge of losing contact with the loop (falling away from the loop), which means $N = 0$. Thus

$$-mg = m\left(-\frac{v^2}{R}\right) \quad \text{and} \quad v = \sqrt{gR}$$

Note that this speed requirement is independent of the mass of the stuntman and his bicycle.

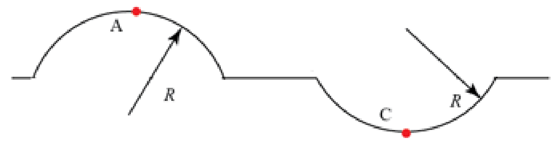
5. Calculations

$$v = \sqrt{gR} = \sqrt{9.8\ m/s^2 \cdot 2.7\ m} = 5.1\ m/s$$

6. Units - correct. We do not have experience with this stunt, but 5.1 m/s (or 11.4 mph) is a reasonable speed for a bike.

<u>Example 5-10</u>

A stuntman drives a car at constant speed as he travels through the hill and valley. The cross sections of both the hill and valley parts can be approximated by a circle of radius 160 m. The mass of the car is 1000 kg, and the mass of the driver is 80 kg.



a) What is the greatest speed (in mph units) at which he can drive without the car leaving the road at the top of the hill (point A)?
b) What is the magnitude of the force on the stuntman (in unit of his weight) at the bottom of the hill (point C)? (note that his speed at point C is the same as at point A).

SOLUTION

1. Physics – circular motion and Newton's laws

2. Equation of motion

$$F_{net} = ma = \frac{mv^2}{R}$$

3 and 4. A free-body diagram at points A and B is quite simple.

At point A the direction of centripetal acceleration is down, then

$$N - mg = -\frac{mv^2}{R}, \qquad \text{and} \quad N = mg - \frac{mv^2}{R}$$

Losing contact with the road means $N = 0$, then the maximum speed at point A is

$$v_{max} = \sqrt{gR}$$

At point B the direction of the centripetal acceleration is up (toward the center of the circle)

$$N - mg = \frac{mv^2}{R}, \qquad \text{and} \quad N = mg + \frac{mv^2}{R}$$

or in units of his weight it will be (keeping in mind that $v = v_{max}$)

$$\text{and} \quad N_w = \frac{N}{mg} = 1 + \frac{v^2}{gR} = 1 + \frac{gR}{gR} = 1 + 1 = 2$$

5. Calculations

$$v_{max} = \sqrt{gR} = \sqrt{9.8\ m/s^2 \cdot 160\ m} = 39.6\ m/s = 89\ mph$$

$$N_w = 2$$

6. Units - correct. The numbers look reasonable.

<u>Example 5-11*</u>

A 1000-kg car rounds a curve on a road of radius 50 meters. The coefficient of friction between the pavement and the car is $\mu_s = 0.60$. What is the maximum possible speed to make the turn without skidding?

a) On a flat road with the coefficient of static friction $\mu_s = 0.60$

b) On a banked icy road with $\beta = 10^0$ and $\mu_s = 0.0$

c) On a banked road with $\beta = 10^0$ and $\mu_s = 0.60$

SOLUTION:

1. Physics – Uniform circular motion with the following forces: gravity, normal, friction

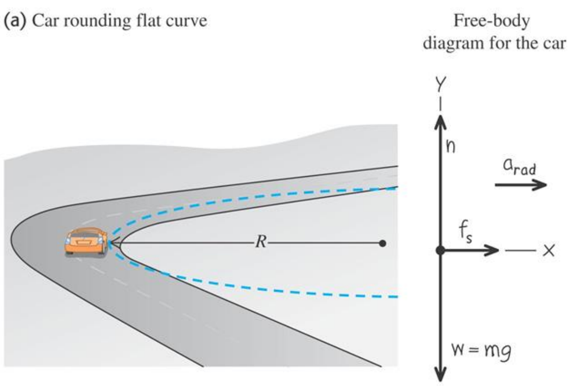
2. The principal equations are

$$F_{net,x} = mv^2/r \qquad f_s = \mu_s n$$

however, the net forces are different for various cases. It is clear that the last case (case c) is the most general, and a) and b) solutions are just special cases

3. For better understanding let's solve a) then b) and then c)

4a) a flat road



$$F_{net,x} = f_s = m\frac{v^2}{R}$$

$$F_{net,y} = n - mg = 0$$

from the second equation

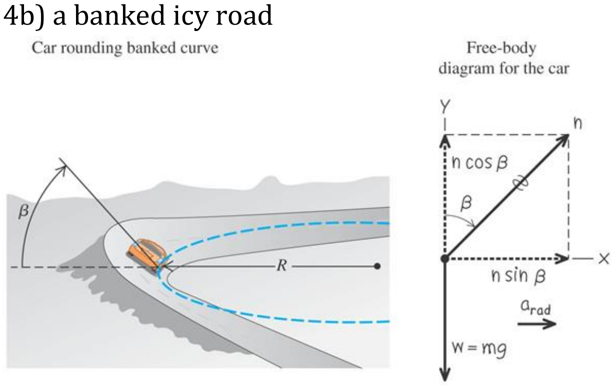
$$n = mg$$

then $f_s = \mu_s n = \mu_s mg$ and

$$\mu_s mg = m\frac{v^2}{R}$$

$$a) \quad v_{max} = \sqrt{\mu_s g R}$$

4b) a banked icy road



$$F_{net,x} = n\sin\beta = m\frac{v^2}{R}$$

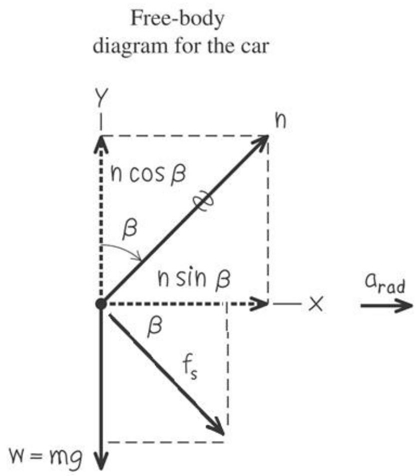
$$F_{net,y} = n\cos\beta - mg = 0$$

from the second equation $n = \frac{mg}{\cos\beta}$ and

$$\frac{mg}{\cos\beta}\sin\beta = m\frac{v^2}{R}$$

$$b)\quad v_{max} = \sqrt{gR\tan\beta}$$

4c) a banked road



Note that we use immediately that $f_s = \mu_s n$

for x-component

$$F_{net,x} = n\sin\beta + \mu_s n\cos\beta = m\frac{v^2}{R}$$

for y-component

$$F_{net,y} = n\cos\beta - \mu_s n\ \sin\beta - mg = 0$$

from the second equation

$$n = \frac{mg}{\cos\beta - \mu_s\sin\beta}$$

the first equation reads

$$n(\sin\beta + \mu_s\cos\beta) = m\frac{v^2}{R}$$

and now

$$\frac{mg}{\cos\beta - \mu_s\sin\beta}(\sin\beta + \mu_s\cos\beta) = m\frac{v^2}{R}$$

$$c)\quad v_{max} = \sqrt{gR\frac{(\sin\beta + \mu_s\cos\beta)}{(\cos\beta - \mu_s\sin\beta)}}$$

Let's do a quick check if we can get answers a) and b) from the last equations

If $\beta = 0$ (flat road) then

$$v_{max} = \sqrt{gR\frac{(\sin\beta + \mu_s\cos\beta)}{(\cos\beta - \mu_s\sin\beta)}} = \sqrt{gR\frac{\mu_s\cos\beta}{\cos\beta}} = \sqrt{\mu_s gR}$$

If $\mu_s = 0$ (no friction) then

$$v_{max} = \sqrt{gR\frac{(\sin\beta + \mu_s\cos\beta)}{(\cos\beta - \mu_s\sin\beta)}} = \sqrt{gR\frac{\sin\beta}{\cos\beta}} = \sqrt{gR\tan\beta}$$

We do get special cases from our general solution c)

5. Calculations

$$a)\ \ v_{max} = \sqrt{\mu_s g R} = 17\ m/s$$

$$b)\ \ v_{max} = \sqrt{gR\tan\beta} = 9.3\ m/s$$

$$c)\ \ v_{max} = \sqrt{gR\frac{(\sin\beta + \mu_s\cos\beta)}{(\cos\beta - \mu_s\sin\beta)}} = 20\ m/s$$

6. How can we validate the numerical answers? Let's compare the max speeds with our experiences. In mph units $v_a = 38\ mph \quad v_b = 21\ mph \quad v_c = 46\ mph$. These speeds look realistic. Besides we can see that a banked road (case c) provides a bit higher max speed comparing to a flat road (case a).

It is interesting how these max speeds would change on an icy road with $\mu_s = 0.1$. In this case we have

$$v_a = 16\ mph \quad v_b = 21\ mph \quad v_c = 26\ mph$$

6 Kinetic Energy, Work, Power

6.1 Energy

Newton's laws of motion give us a tool to analyze and predict varieties of motion. However, the analysis is often complicated. We may need to solve numerically a set of differential equations based on Newton's laws. Or finding a solution may require details that we do not know.

There is another, very powerful, technique for analyzing motion based on conservation of energy. We often use this technique when we are not interested in some details but care about only initial and final states or configurations. On the other hand, energy comes in many different forms. Therefore, using conservation of energy can be a delicate issue, even for relatively simple systems.

Even though we extensively use the word *energy* in everyday life, there is no precise definition for it. Richard Feynman (Nobel Prize in physics) wrote "In physics today, we have no knowledge of what energy is. We know how to calculate its value for a great variety of situations, but beyond that it's just an abstract thing which has only one really important property – conservation".

6.2 Kinetic Energy

Let's start our consideration with *kinetic energy*, namely energy associated with the state of motion of an object. For an object of mass *m* travelling with speed *v* we define kinetic energy as

$$K = \frac{1}{2}mv^2 \quad (kinetic\ energy)$$

Energy is a scalar quantity (a number) that is associated with a state (or condition) of one or more objects.

The SI unit of kinetic energy as well as any other type of energy is the *joule* (J)

$$1 \text{ joule} = 1 \text{ J} = 1 \text{ kg m}^2/\text{s}^2.$$

Let's find out a change in kinetic energy for a particle travelling with constant acceleration along x-coordinate

$$K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}m\left(v_f^2 - v_i^2\right)$$

Using $v_f^2 = v_i^2 + 2a\left(x_f - x_i\right)$ from chapter 2 we can rewrite

$$K_f - K_i = \frac{1}{2}m2a\left(x_f - x_i\right) = ma\left(x_f - x_i\right) = F\left(x_f - x_i\right)$$

Thus we have a connection between the change in kinetic energy and a force causing this change.

It is interesting to consider a more general case, namely a motion with a variable force in two or three dimensions. However, before doing that we need to step back and consider a scalar product of two vectors.

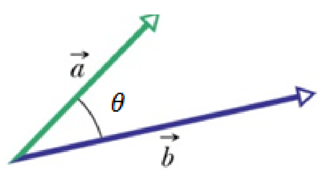
6.3 Scalar (dot) product of vectors

There are two kinds of product of a vectors but none of them is like a common algebraic multiplication. The first kind is called the scalar (or dot) product. It produces a result that is a scalar quantity. The second kind is called the vector (or cross) product. It yields a new vector.

The scalar product of the vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \cdot \vec{b}$ and defined to be

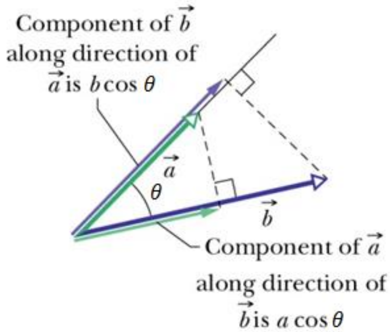
$$\vec{a} \cdot \vec{b} = ab\cos\theta \tag{6.1}$$

here a is the magnitude of $\vec{a}$, b is the magnitude of $\vec{b}$, and θ is the angle between the directions of $\vec{a}$ and $\vec{b}$. There are actually two such angles: θ and $360 - \theta$. Either can be used because their cosines are the same.



The scalar product can be rewritten as

$$\vec{a} \cdot \vec{b} = ab\cos\theta = (a\cos\theta)b = a(b\cos\theta)$$



The commutative law applies to a scalar product, so we can write

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

The scalar product involving unit vectors $\hat{x}$ and $\hat{y}$ (here we use $\hat{x}$ and $\hat{y}$ for unit vectors notations instead of $\hat{\imath}$ and $\hat{\jmath}$.)

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = (1)(1)\cos 0 = 1$$
$$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = (1)(1)\cos 90 = 0$$

Then a scalar product of a unit vector $\hat{x}$ and a vector $\hat{a}$

$$\hat{x}\cdot\vec{a} = \hat{x}\cdot\left(a_x\hat{x} + a_y\hat{y}\right) = a_x\hat{x}\cdot\hat{x} + a_y\hat{x}\cdot\hat{y} = a_x$$

gives a projection a_x of $\vec{a}$ onto this unit vector $\hat{x}$.

Let's calculate a scalar product of two vectors $\vec{a}$ and $\vec{b}$ using vector components

$$\vec{a}\cdot\vec{b} = \vec{a}\cdot\left(b_x\hat{x} + b_y\hat{y}\right) = \vec{a}\cdot\hat{x}\,b_x + \vec{a}\cdot\hat{y}\,b_y = a_xb_x + a_yb_y$$

Thus, we can write the scalar product in two forms

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$
$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y \tag{6.2}$$

Note that for $\vec{a} = \vec{b}$

$$\vec{a}\cdot\vec{a} = a_xa_x + a_ya_y = a_x^2 + a_y^2 = a^2 \tag{6.3}$$

6.4 Kinetic energy and work

Let us imagine the particle moving through space between two points $\hat{r}_i$ and $\hat{r}_f$. The time derivative of kinetic energy can be easily evaluated if we use $\vec{v}\cdot\vec{v} = v^2$

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\frac{d}{dt}(v^2) = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\frac{d\vec{v}}{dt}\cdot\vec{v} + \frac{1}{2}m\vec{v}\cdot\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dt}\cdot\vec{v} = \vec{F}\cdot\vec{v}$$
$$= \vec{F}\cdot\frac{d\vec{r}}{dt}$$

If we multiply both sides by dt we find

$$dK = \vec{F}\cdot d\vec{r}$$

Integrating both sides along a path connecting points $\hat{r}_i$ and $\hat{r}_f$ gives the Kinetic Energy – Work theorem.

$$\Delta K = K_f - K_i = \int_i^f \vec{F}\cdot d\vec{r} = W(i\to f) \tag{6.4}$$

where $W(i \to f)$ is *the work*[2] done by force $\vec{F}$ moving from point i to point f.

In evaluating a path integral, like the integral above, it is usually possible to convert it into a regular integral over a single variable by choosing an appropriate coordinate system, or replace it on a sum of single variable integrals using vector components, namely

$$\vec{F} = F_x\,\hat{x} + F_y\,\hat{y}$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

then

$$\vec{F} \cdot d\vec{r} = \left(F_x\,\hat{x} + F_y\,\hat{y}\right) \cdot (dx\,\hat{x} + dy\,\hat{y}) = F_x dx + F_y dy$$

and finally

$$W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy. \tag{6.5}$$

If there are two (or more) forces, then

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

$$W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_i^f \vec{F}_1 \cdot d\vec{r} + \int_i^f \vec{F}_2 \cdot d\vec{r}.$$

In case of constant force acting along a linear path (let's say along x) can be easily written as

$$W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x dx = \int_{x_i}^{x_f} F \cos\theta\, dx = F \cos\theta \left(x_f - x_i\right) = Fd\cos\theta = \vec{F} \cdot \vec{d}$$

or

$$W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta \tag{6.6}$$

In many textbooks this is the most common definition for work done by a constant force along a line.

Using the vector component form the same result can be written as

$$W(i \to f) = F_x\left(x_f - x_i\right) + F_y(y_f - y_i) \tag{6.7}$$

For work we use the same unit joule (J) as for energy: 1 joule = 1 J = 1 kg·m²/s² = 1 N·m.

[2] "The word "work" in physics has a meaning so different from that of the word as it is used in ordinary circumstances that it must be observed carefully that there are some peculiar circumstances in which it appears not to be the same. For example, according to the physical definition of work, if one holds a hundred-pound weight off the ground for a while, he is doing no work. Nevertheless, everyone knows that he begins to sweat, shake, and breathe harder, as if he were running up a flight of stairs. Yet running upstairs is considered as doing work (in running downstairs, one gets work out of the world, according to physics), but in simply holding an object in a fixed position, no work is done. Clearly, the physical definition of work differs from the physiological" R. Feynman

6.5 Power

A given amount of work W may be done either in a short time or a long time. If an external force is applied to an object, and if the work done by this force in the time interval Δt is W, then *the average power* is defined as

$$P_{avg} = \frac{W}{\Delta t} \tag{6.8}$$

The instantaneous power is defined as the limiting value of the average power as Δt approaches zero

$$P = \frac{dW}{dt} \tag{6.9}$$

Another definition can be derived from $dW = F\cos\theta\, dx$, namely

$$P = \frac{dW}{dt} = \frac{F\cos\theta\, dx}{dt} = F\cos\theta\frac{dx}{dt} = Fv\cos\theta = \vec{F}\cdot\vec{v}$$

or instantaneous power

$$P = \vec{F}\cdot\vec{v} \tag{6.10}$$

Note that we got this result by differentiating kinetic energy as

$$\frac{dK}{dt} = \vec{F}\cdot\vec{v} = \vec{F}\cdot\frac{d\vec{r}}{dt}.$$

SI unit for power is 1 watt = 1 W = 1 J/s. Other popular unit is 1 horsepower = 1 hp = 746 W.

Note that energy (and work) can be measured in watt per second, or most commonly 1 kilowatt-hour = 1 kW * h = $3.60*10^6$ J = 3.60 MJ.

6.6 Examples

Note that calculations for work done by various forces are quite straightforward in university physics courses. As a rule all forces (but spring one) are constant forces in introductory physics classes. Therefore the primary equation is

$$W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

The result can be positive or negative depending on the angle θ. Choosing a proper coordinate system makes a difference. One may find it is easier to use the component form

$$W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i)$$

Note that components F_x and F_y can be positive of negative.

While integrating using components

$$W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy$$

can be fun to do, it is unnecessary unless dealing with variable forces like the spring force $F = -kx$ (Hooke's law). For the spring force

$$W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} (-kx)dx = -\frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

where the origin is always placed at $x = 0$ (where the spring is in its relaxed state).

In case of a couple forces acting on an object one may use

$$W(i \to f) = \sum_{j=1}^{n} W_j(i \to f).$$

<u>Example 6-1</u>

A tennis player hits a 58.0-g tennis ball so that it goes straight up and reaches a maximum height of 8.0 m. How much work does the gravity do on the ball on the way up? On the way down?

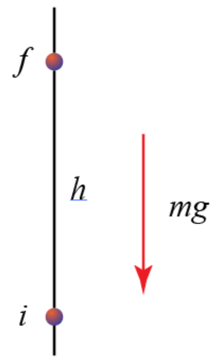
SOLUTION:

1. Physics – work and forces

2. The basic equations that we may use

$$W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta \quad \text{or} \quad W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i)$$

We are free to choose what form to use. I prefer the component form but since the form $W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta$ is a standard one in many textbook I'll keep using the both forms and you decide for yourself which one do you like more.



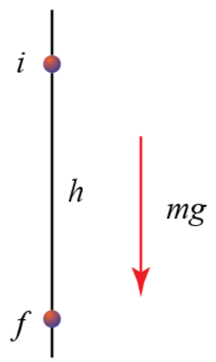
3. Having a diagram helps to see the directions and coordinates. From the diagram it is clear that the force of gravity has only one component, namely $F_y = -mg$

4. Then on the way up

$W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta = mgh\cos 180^0 = -mgh$

If we use the component form,

$W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i) = 0 - mg(y_f - y_i) = -mgh$



The way down is very straightforward.

$W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta = mgh\cos 0^0 = mgh$

$W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i) = 0 - mg(y_f - y_i) = -mg(-h) = mgh$

5. Calculations are very simple

W_{up} = -0.058 kg * 9.8 m/s2 * 8.0 m = -4.5 J

W_{down} = -W_{up} = 4.5 J

6. We do have proper dimension (joules). As for the number it does not related well to our everyday experience. Usually we do not use *joules* in day-to-day life.

Example 6-2

A 5.00-kg package slides 1.5 m down a long ramp that is inclined at 12^0 below horizontal. The coefficient of kinetic friction between the package and the ramp is $\mu_k = 0.31$. Calculate

a) the work done on the package by friction
b) the work done on the package by the gravity
c) the work done by the normal force
d) the total work done on the package
e) If the package had has a speed of 2.20 m/s at the top of the ramp, what is its speed after sliding 1.5 m down the ramp?

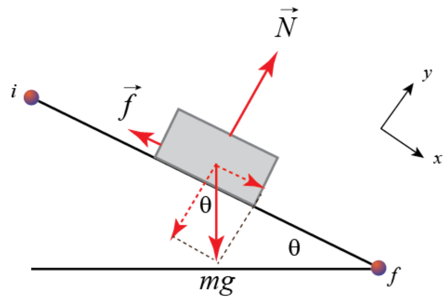
SOLUTION:

1. Physics – work and forces, kinetic energy-work theorem

2. The basic equations that we may use

$$W(i \to f) = \vec{F} \cdot \vec{d} = Fd\cos\theta \quad \text{or} \quad W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i)$$

$$\Delta K = K_f - K_i = \int_i^f \vec{F} \cdot d\vec{r} = W(i \to f)$$



3. Having a proper diagram would help to avoid confusion with directions and angles.

Let's choose the coordinate system such as the incline surface is along x-coordinate. It makes easier to evaluate the work along the surface.

For every force we have following components

Friction: $f_x = -\mu N_y \quad f_y = 0$

Gravity: $F_{gx} = mg \sin\theta \qquad F_{gy} = -mg\cos\theta$

Normal: $N_x = 0 \quad N_y = -F_{gy} = mg\cos\theta$

(Note that you may consider using Newton's second law to find a connection between the forces if you are not sure in the above force components).

4. Let's use the component form for the work

$W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i)$

with a notation $d = x_f - x_i$ and also note that $y_f - y_i = 0$ in our coordinate system.

The work done by friction $W_f = F_x(x_f - x_i) + F_y(y_f - y_i) = -\mu mg\cos\theta\, d + 0 \cdot 0 = -\mu mgd\cos\theta$

The work done by gravity $W_g = F_x(x_f - x_i) + F_y(y_f - y_i) = mg\sin\theta\, d - mg\cos\theta \cdot 0 = mgd\sin\theta$

The work done by normal force $W_N = F_x(x_f - x_i) + F_y(y_f - y_i) = 0 \cdot d + mg\cos\theta \cdot 0 = 0$

The total work done by all forces is $W = W_f + W_g + W_N$

The speed at the end of the ramp

$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W \quad \text{or} \quad v_f = \sqrt{v_i^2 + 2W/m}$

5. Calculations

$W_f = -0.31 \cdot 5\,kg \cdot 9.8\ m/s^2 \cdot 1.5\,m \cdot \cos 12^0 = -22.3\,J$

$W_g = 5\,kg \cdot 9.8\ m/s^2 \cdot 1.5\,m \cdot \sin 12^0 = 15.3\,J$

$W_n = 0\,J$

$W = W_f + W_g + W_N = -7.0\,J$

$v_f = \sqrt{(2.2\ m/s)^2 - 2 \cdot 7\,J/5\,kg} = 1.4\ m/s$

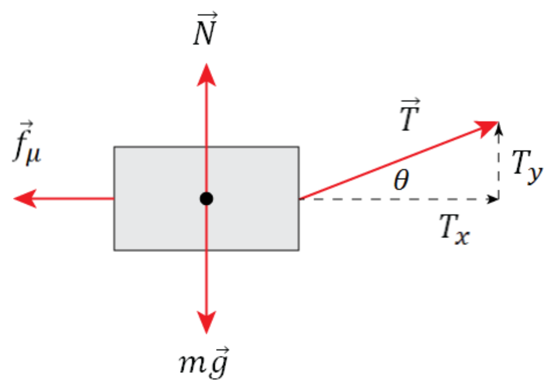
6. We have correct dimensions for the work. Let's check the last term for the speed (under the square root) [J]/[kg] = kg·m^2/s^2/kg = m^2/s^2 that is v^2 (correct).

We may also notice that the final speed is less that the initial speed. It corresponds to a case when the frictional force is larger than gravity along the ramp thus doing more work.

Example 6-3

A 6.0 kg cat sleeps on a mat. A dog pulls the mat across the floor using a rope that makes 20^0 above the floor. The tension is a constant 20.0 N and the coefficient of friction is 0.20. Find cat's speed after being pulled 2.0 m.

SOLUTION:



1. Physics –kinetic energy-work theorem

2. The basic equations that we may use

$$W(i \to f) = F_x(x_f - x_i) + F_y(y_f - y_i)$$

$$K_f - K_i = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = W(i \to f)$$

3. The free-body diagram on the right has all the forces in place. For every force we have following components

Tension: $T_x = T\cos\theta$, $T_y = T\sin\theta$

Gravity: $F_{gx} = -mg$, $F_{gy} = 0$

Normal force: $N_x = 0$, N_y

Friction: $f_{\mu x} = -\mu N_y$, $f_{\mu y} = 0$

4. We work only with the x −components because there is no displacement in the y − direction.

First we need to find the normal force from second newton's law for

First we need to find the normal force from second newton's law for y −components

$$N_y + T\sin\theta - mg = 0, \quad N_y = mg - T\sin\theta, \quad f_{\mu x} = -\mu(mg - T\sin\theta)$$

now

$$\frac{mv_f^2}{2} = \big(T\cos\theta - \mu(mg - T\sin\theta)\big)x_f, \qquad v_f^2 = \frac{2}{m}\big(T\cos\theta - \mu(mg - T\sin\theta)\big)x_f$$

5. Calculations

$$v_f = 2.4\, m/s$$

6. The answer seems convincing.

<u>Example 6-4</u>

Your job is to lift 30-kg crates a vertical distance of 0.90 m from the ground onto the bed of a truck. How many crates you have to load onto the truck in 1 minute for the average power output you use to lift the crates to equal 1.0 hp? (Working like a horse!)

SOLUTION:

1. Physics – power, work and forces

2. Basic equations: we need average power (because we speak about relatively large time intervals) as well work done by you against gravity

$P_{avg} = \dfrac{W}{\Delta t}$ (average power) $\qquad W_{you} = -(-mgh) = mgh$

3. Having the power and time we may calculate how much work can be done with this power. Dividing this work by the work to lift one box we can find how many boxes we should lift in one minute.

4. Total work with given power $W = P_{avg}\Delta t$, and the work to lift one box $W_{box} = mgh$

$$N_{boxes} = \frac{W}{W_{box}}$$

5. Calculations

First we evaluate power in watt

$1.0\ hp = 1.0\ hp \dfrac{746\ W}{1\ hp} = 746\ W$

$N = \dfrac{746\ W \cdot\ 60\ s}{30\ kg \cdot 9.8\ \ m/s^2 \cdot 0.9\ m} = 169$ boxes

6. Let's check the dimensions. The numerator [W]*[s] = [J] = $kg{\cdot}m^2/s^2$, the denominator is $kg{\cdot}m^2/s^2$ (OK)

It is clear that we cannot load so many heavy boxes in one minute, thus we cannot work like a horse!

7 Conservation of Energy

7.1 Potential energy and conservative forces

In nature there are certain forces which have a very remarkable property which we call "conservative" (no political ideas involved). If we calculate how much work is done by a force in moving an object from one point to another along some curved path, in general the work depends upon the path; but, in special cases it does not. If it does not depend upon the path, we say that the force is a conservative force.

Strictly speaking there are two conditions for a force to be conservative. A force $\vec{F}$ acting on a particle is conservative if and only if it satisfy two conditions:

1. $\vec{F}$ depends only on the particle's position $\vec{r}$ (and not on the velocity $\vec{v}$, or the time t, or any other variable) that is $\vec{F} = \vec{F}(\vec{r})$.
2. For any two points $\vec{r}_1$ and $\vec{r}_2$, the integral

   $$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

   is the same for all paths between points 1 and 2, or work $W(1 \to 2)$ done by $\vec{F}$ is independent of paths between points 1 and 2.

The reason for the name conservative and for the importance of the concept is this: if all forces acting on an object are conservative, we can define a quantity called the potential energy, denoted $U(\vec{r})$, a function of only position, with the property that the total mechanical energy

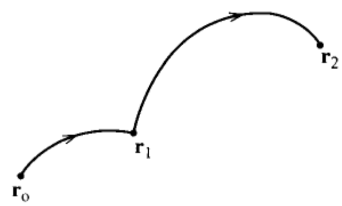
$$E = K + U(\vec{r}) \tag{7.1}$$

is conserved.

To define the potential energy $U(\vec{r})$ corresponding to a given conservative force, we first choose a reference point $\vec{r}_0$ at which U is defined to be zero (e.g. in the case of gravity near the earth's surface, we often define U to be zero at ground level.) We then define $U(\vec{r})$, the potential energy at an arbitrary point $\vec{r}$, to be

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} F(\vec{r}')d\vec{r}' = -W(\vec{r}_0 \to \vec{r}) \tag{7.2}$$

In words, $U(\vec{r})$ is minus the work done by $\vec{F}$ if the particle moves from the reference point $\vec{r}_0$ to the point of interest $\vec{r}$. Notice that the definition above only makes sense because of the property (2) of conservative forces. If the integral were different for different paths, it would not define a unique function $U(\vec{r})$.



We can now derive a crucial expression for the work done by $\vec{F}$ in terms of the potential energy $U(\vec{r})$. Let $\vec{r}_1$ and $\vec{r}_2$ be any two points as in Figure on the right. If $\vec{r}_0$ is the reference point at which U is zero, then it is clear from that

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2)$$

and hence

$$W(\vec{r}_1 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_2) - W(\vec{r}_0 \to \vec{r}_1)$$

Each of the two terms on the right is (minus) the potential energy at the corresponding point. Thus we have proved that the work on the left is just the difference of these two potential energies:

$$W(\vec{r}_1 \to \vec{r}_2) = -[U(\vec{r}_2) - U(\vec{r}_1)] = -\Delta U$$

The usefulness of this result emerges when we combine it with the Kinetic Energy – Work theorem

$$\Delta K = K_2 - K_1 = \int_1^2 \vec{F} \cdot d\vec{r} = W(1 \to 2)$$

Namely we can now write

$$\Delta K = -\Delta U \tag{7.3}$$

or, moving the right side across to the left

$$\Delta(K + U) = 0$$

That is, the mechanical energy

$$E = K + U \tag{7.4}$$

does not change as the particle moves from $\vec{r}_1$ to $\vec{r}_2$. Since the points $\vec{r}_1$ and $\vec{r}_2$ were any two points on the particle's trajectory, we have the important conclusion: If the force on a particle is conservative, then the particle's mechanical energy never changes; that is, the particle's energy is conserved, which explains the use of the adjective "conservative."

So far we have established the conservation of energy for a particle subject to a single conservative force. If the particle is subject to several forces, all of them conservative, our result generalizes easily.

Principle of Conservation of Energy for One Particle

If all of the n forces $\vec{F}_i$ $(i = 1, \ldots, n)$ acting on a particle are conservative, each with its corresponding potential energy $U_i(\vec{r})$, the **total mechanical energy**, defined as

$$E = K + U_1(\vec{r}) + U_2(\vec{r}) + \cdots + U_n(\vec{r}) \tag{7.5}$$

is constant in time.

7.1.1 Force as a gradient of potential energy

We have seen that the potential energy $U\ (r)$ corresponding to a force $\vec{F}$ can be expressed as an integral of $\vec{F}$. This suggests that we should be able to write $\vec{F}$ as some kind of derivative of $U\ (r)$. Using some mathematics we can show that

$$\vec{F} = -\vec{\nabla} U = -\hat{x}\frac{\partial U}{\partial x} - \hat{y}\frac{\partial U}{\partial y} - \hat{z}\frac{\partial U}{\partial z} \tag{7.6}$$

This expression gives formal definition of potential energy U. In one-dimensional case, this connection between force and potential energy is particularly simple:

$$F_x = -\frac{dU}{dx} \tag{7.7}$$

Let's note that U is always arbitrary to within an additive constant, because taking a derivative of a constant function gives zero:

$$\vec{\nabla}(U + c) = \vec{\nabla} U + \vec{\nabla} c = \vec{\nabla} U$$

Therefore, the force $\vec{F}$ (which is an "explicitly observable" quantity) does not change if we add a constant to the "auxiliary" quantity U. In other words, the potential energy U gives rise to the same force as the potential energy $U + c$. Since it is the force (rather than the potential energy) that is physically observable, we say that these two potential energies are equivalent.

It is not the absolute value of potential energy that is important. Rather, it is the change in potential energy over distance (the gradient) that creates force and is, therefore, important.

A consequence of this observation is that, for a given problem, the "zero" of potential can be chosen arbitrarily as a matter of convenience.

7.2 Gravitational and elastic potential energies

In University Physics I we usually deal with two conservative forces, namely gravity and elastic (spring) force. In University Physics II we add the Coulomb force on electric charge.

7.2.1 Gravitational potential energy

If we are not going to heights comparable with the radius of the earth, then the force of gravity is a constant vertical force $F_y = -mg$. Then

$$F_y = -mg = -\frac{dU_g}{dy}$$

$$dU_g = mgdy, \quad \text{then} \quad \int dU_g = \int_0^y mgdy \qquad \text{and finally}$$

$$U_g(y) = mgy \tag{7.8}$$

Let's consider a change in gravitational potential energy between two points using the definition of the work based on the integral from a force[3]. Since there is only one y-component

$$\Delta U_g = U_{gf} - U_{gi} = -\int_{y_i}^{y_f} F_y dy = \int_{y_i}^{y_f} mgdy = mg(y_f - y_i)$$

$$\Delta U_g = mg(y_f - y_i) \tag{7.9}$$

As we noted above, only changes in potential energy are physically meaningful. However to simplify calculations we may select a reference point y_i where $U_i = 0$, then $U(y) = mgy$.

Good to remember: The gravitational potential energy associated with a particle-Earth system depends ONLY on the vertical position y (or height) of the particle relative to the reference position ($y = 0$), not on the horizontal position. And again, only differences in potential energy count.

7.2.2 Elastic (spring) potential energy

For the elastic restoring force of a spring $F = -kx$ is a good approximation for many springs (Hooke's law) where k is a spring constant (a SI unit for k is N/m).

$$F_x = -kx \quad \Rightarrow \quad -\frac{dU_s}{dx} = -kx$$

$$dU_s = kxdx \quad \Rightarrow \quad \int dU_s = \int_0^x kxdx$$

then

---

[3] from chapter 6: $W(i \to f) = \int_i^f \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy$. Since work and the potential energy are connected $W(i \to f) = -[U(\vec{r}_f) - U(\vec{r}_i)] = -\Delta U$, then $\Delta U = -\int_{x_i}^{x_f} F_x dx - \int_{y_i}^{y_f} F_y dy$

$$U_s = \frac{1}{2}kx^2 \tag{7.10}$$

The zero of potential energy is at the point $x = 0$, the equilibrium position of the spring. Again we could add any constant we wish to U_s but not to x.

Now we do the same using $\Delta U = -\int_{x_i}^{x_f} F_x dx - \int_{y_i}^{y_f} F_y dy$

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x dx = \int_{x_i}^{x_f} kxdx = k\int_{x_i}^{x_f} xdx = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2$$

$$\Delta U_s = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \tag{7.11}$$

Important: *The reference point for the spring potential energy must be the equilibrium point.*

7.3 Non-conservative forces

As a matter of fact, all the fundamental forces in nature appear to be conservative. This is not a consequence of Newton's laws. In fact, so far as Newton himself knew, the forces could be nonconservative, as friction apparently is. When we say friction apparently is, we are taking a modern view, in which it has been discovered that all the deep forces, the forces between the particles at the most fundamental level, are conservative.

When we study matter in the finest detail at the atomic level, it is not always easy to separate the total energy of a thing into two parts, kinetic energy and potential energy, and such separation is not always necessary. In many situations, it is practical to replace vast amount of conservative forces between particles on a single nonconservative force. Then all these different forms of internal energy are sometimes considered as "lost". When friction is present it is not true that kinetic energy is lost, even though a sliding object stops and the kinetic energy seems to be lost. The kinetic energy is not lost because, of course, the atoms inside are jiggling with a greater amount of kinetic energy than before, and although we cannot see that, we can measure it by determining the temperature. Of course if we disregard the heat energy, then the conservation of energy theorem will appear to be false.

If some of the forces on our particle are nonconservative (like friction), then we cannot define corresponding potential energies; nor can we define a conserved mechanical energy.

Nevertheless, we can define potential energies for all of the forces that are conservative, and then recast the Kinetic Energy – Work theorem in a form that shows how the nonconservative forces change the particle's mechanical energy. First, we divide the net force on the particle into two parts, the conservative part $\vec{F}_{cons}$ and the nonconservative part $\vec{F}_{nc}$. For conservative forces we can define a potential energy, which we'll call just U. By the Kinetic Energy – Work theorem, the change in kinetic energy between any two times is

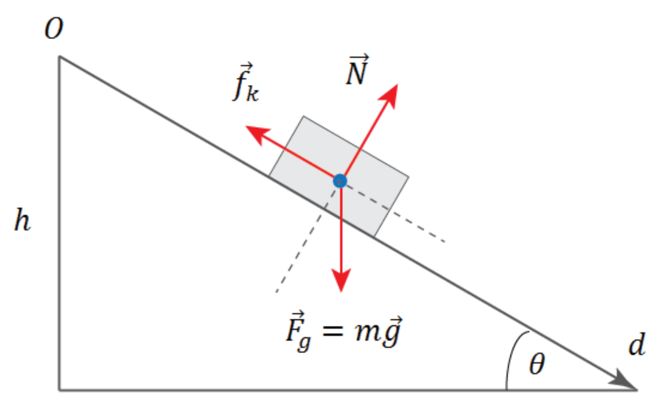
$$\Delta K = W_{cons} + W_{nc}$$

The first term on the right is just $-\Delta U$ and can be moved to the left side to give $\Delta(K+U) = W_{nc}$. If we define the mechanical energy as $E = K + U$, then we see that $\Delta E = \Delta(K+U) = W_{nc}$ or

$$K_i + U_i = K_f + U_f - W_{nc} \tag{7.12}$$

Mechanical energy is no longer conserved, but we have the next best thing. The mechanical energy changes to precisely the extent that the nonconservative forces do work on our particle. In many problems, the only nonconservative force is the force of sliding friction, which usually does negative work. (The frictional force f is in the direction opposite to the motion, so the work done by a frictional force is negative.) In this case W_{nc} is negative and the object loses mechanical energy in the amount "stolen" by friction $W_{nc} = W_{fr} = -f_\mu\, d$.

Let's consider as an example a block sliding down an incline, namely a block of mass m accelerating from rest down incline that has a coefficient of friction μ and is at angle θ from horizontal. Let's find its speed v when it reaches the bottom of the slope, a distance d from its starting point O.



The setup and the forces on the block are shown in Figure. The three forces on the block are its weight, $\vec{w} = \vec{F}_g = m\vec{g}$, the normal force of the incline, $\vec{N}$, and the frictional force $\vec{f}$, whose magnitude to be $f_\mu = \mu mg\cos\theta$. The force of gravity is conservative, and the corresponding potential energy is $U = mgy$ where y is the block's vertical height above the bottom of the slope (if we choose the zero of potential energy at the bottom). The normal force does no work, since it is perpendicular to the direction of motion, so will not contribute to the energy balance. The frictional force does work $W_{fr} = -fd = -\mu mgd\cos\theta$. The change in kinetic energy is

$$K_f - K_i = \frac{1}{2}mv^2$$

and the change in potential energy is

$$\Delta U = U_f - U_i = -mgh = -mgd\sin\theta$$

Then

$$\Delta E = \Delta(K+U) = W_{nc}$$

$$\frac{1}{2}mv^2 - mgd\sin\theta = -\mu mgd\cos\theta$$

Solving for v we find

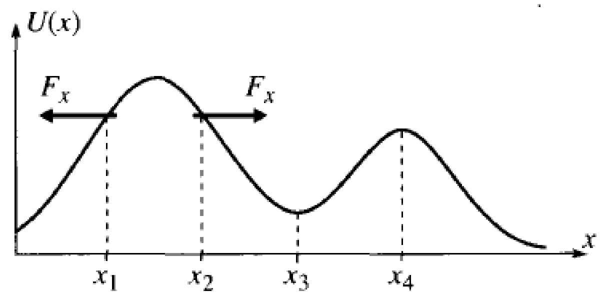
$$v = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

As usual, you should check that this answer agrees with common sense. For example, does it give the expected answer when $\theta = 90^0$? What about $\theta = 0^0$?

7.4 Potential energy diagrams

A useful feature of one-dimensional systems is that with only one independent variable (x) we can plot the potential energy $U(x)$, and, as we shall see, this makes it easy to visualize the behavior of the system.

If we plot the potential energy against x as in Figure, we can easily see qualitatively how the object has to behave. The direction of the net force is given by

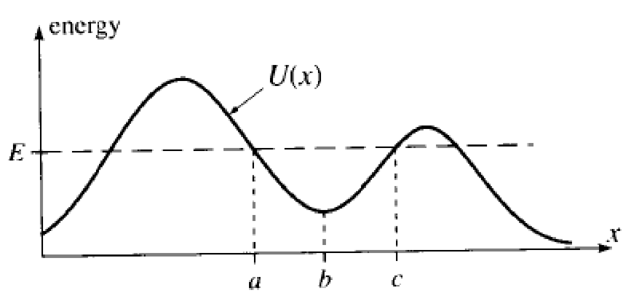


$$F_x = -\frac{dU}{dx}$$

as "downhill" on the graph of $U(x)$ - to the left at x_1 and to the right at x_2 It follows that the object always accelerates in the "downhill" direction - a property that reminds one of the motion of a roller coaster, which also always accelerates downhill. This analogy is not an accident: For a roller coaster, $U(x)$ is mgh (where h is the height above ground) and the graph of $U(x)$ against x has the same shape as a graph of h against x, which is just a picture of the track. For any one-dimensional system, we can always think about the graph of $U(x)$ as a picture of a roller coaster, and common sense will generally tell us the kind of motion that is possible at different places

At points, such as x_3 andx_4, where $dU/dx = 0$ and $U(x)$ is minimum or maximum, the net force is zero, and the object can remain in equilibrium. That is, the condition $dU/dx = 0$ characterizes points of equilibrium. At x_3, where $d^2U/dx^2 > 0$ and $U(x)$ is minimum, a small displacement from equilibrium causes a force which pushes the object back to equilibrium (back to the left on the right of x_3, back to the right on the left of x_3). In other words, equilibrium points where $d^2U/dx^2 > 0$ and $U(x)$ is minimum are points of stable equilibrium. At equilibrium points like x_4 where $d^2U/dx^2 < 0$ and $U(x)$is maximum, a small displacement leads to a force away from equilibrium, and the equilibrium is unstable.

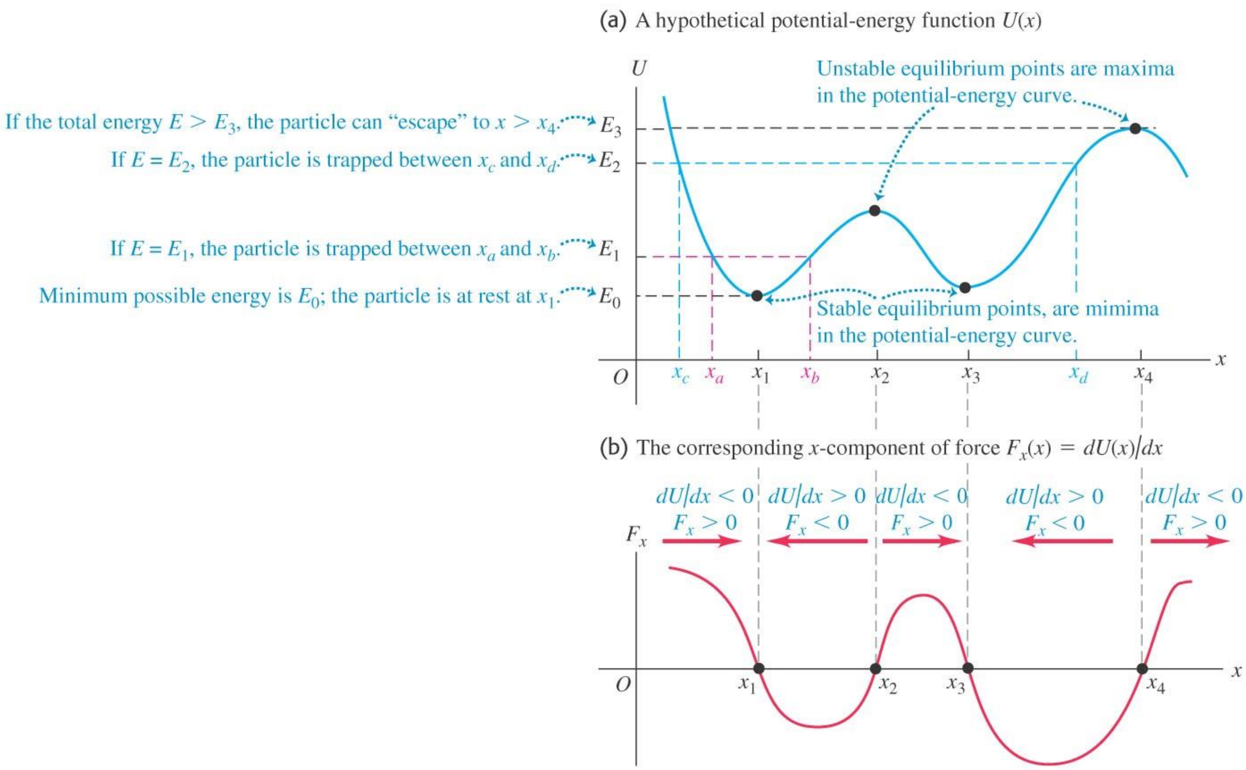
If the object is moving then its kinetic energy is positive and its total energy is necessarily greater than $U(x)$. For example, suppose the object is moving somewhere near the equilibrium point $= b$.



Its total energy has to be greater than $U(b)$ and could, for example, equal the value shown as E in that figure. If the object happens to be on the right of b and moving toward the right, its potential energy will increase and its kinetic energy must therefore decrease until the object reaches the turning point labeled c, where $U(c) = E$ and the kinetic energy is zero. At $x = c$ the object stops and, with the force back to the left, it accelerates back toward $x = b$. It cannot now stop until once again the Kinetic energy is zero, and this occurs at the turning point a, where $U(a) = E$ and the object accelerates back to the right. Since the whole cycle now repeats itself, we see that if the object starts out between two hills and its energy is lower than the crest of both hills, then the object is trapped in the valley or "well" and oscillates indefinitely between the two turning points where $U(x) = E$.

Suppose the cart again starts out between the two hills but with energy higher than the crest of the right hill though still lower than the left. In this case, it will escape to the right since $E > U(x)$ everywhere on the right, and it can never stop once it is moving in that direction. Finally, if the energy is higher than both hills, the cart can escape in either direction.

These considerations play an important role in many fields.



7.5 Guidelines for solving most common problems in “Conservation of energy”

All problems can be divided into two groups: 1) “pure conservation of energy” problems, when one only need to use conservation of energy to find an answer, and 2) “ combined” problems, that may also include 1D motion or 2D projectile motion, or circular motion (pendulums, loops).

Here is a reference table for equations.

Energy	**Equation**	**Comments**
The kinetic energy	$K = \frac{1}{2}mv^2$	
The gravitational potential energy (particle-Earth)	$U_g(y) = mgy$	if the reference frame is set as $y_i = 0$
The elastic potential energy	$U_s(x) = \frac{1}{2}kx^2$	where the reference is set at relaxed length $x = 0$
The total mechanical energy	$K_i + U_i = K_f + U_f$	
The conservation of energy	$K_i + U_i = K_f + U_f + f_k d$	(with frictional force)

Let’s concentrate on applying conservation of energy to a single object. Unlike dealing with forces, the procedure is rather simple for solving problems in Physics I.

Point 1: You start with writing down the law of conservation of energy in most detailed form

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}kx_f^2 + f_k d$$

Attention:

- While you may choose any reference point for the gravitational potential energy, the reference point for the spring potential energy MUST be the equilibrium point.
- If you have a spring, its orientation can be horizontal (then we use $(1/2)kx^2$), or vertical (at that time we use $(1/2)ky_s^2$), or under some angle (for example, a spring on an incline).
- Most mistakes students do in problems with both gravitational and spring energies. *The cause of mistakes is not choosing properly reference points for the two energies*.

Point 2: You analyze what energy you have in your problem, and keep only relevant terms in the above equation.

Point 3: Finally you have one equation with one unknown. That is a simple algebra task.

In case of two connected objects you include all energies for both objects into a single equation (we use conservation of energy for a system). For clarity let’s keep only kinetic and gravitational potential energy (you may easily add the spring energy and energy lost to friction when needed)

$$\frac{1}{2}mv_{1i}^2 + mgy_{1i} + \frac{1}{2}mv_{2i}^2 + mgy_{2i} = \frac{1}{2}mv_{1f}^2 + mgy_{1f} + \frac{1}{2}mv_{2f}^2 + mgy_{2f}$$

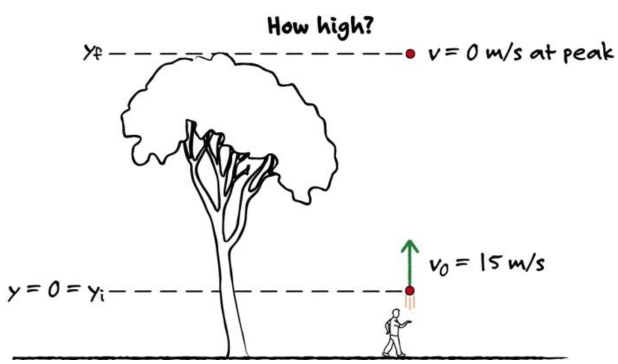
7.6 Examples

Example 7-1

You throw a tennis ball straight up with initial speed $v_i = 15.0\ m/s$. How high does it go above the point where you release it? Ignore air resistance

SOLUTION:

1. Physics – gravity, motion, conservation of energy



2. The basic equation

$$K_i + U_i = K_f + U_f + f_k d$$

3. In this problem there is only one force affecting the flight, namely gravity, therefore we can write

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

with the given conditions $v_f = 0$ and choosing the reference point for gravity as $y_i = 0$

$$\frac{1}{2}mv_i^2 + 0 = 0 + mgy_f$$

4. The equation above produces an instant solution for the final vertical position

$$y_f = \frac{v_i^2}{2g}$$

5. Calculations

$$y_f = \frac{(15.0\ m/s)^2}{2 \cdot 9.8\ m/s^2} = 11.5\ m$$

6. We have got a proper dimension (meters). The numerical answer seems reasonable.

Example 7-2

You are driving at 55 mph when the road suddenly descends 90 ft into a valley. You take your foot off the accelerator and coast down the hill. Just as you reach the bottom you see the policeman hiding behind the speed limit sign that reads “70 mph”. Are you going to get a speeding ticket? (Neglect air and rolling resistance)

SOLUTION:

1. Physics – gravity, motion, conservation of mechanical energy

2. The basic equation

$$K_i + U_i = K_f + U_f$$

3. Then

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

Choosing the reference point for gravity as $y_f = 0$ (at the sign 70 mph)

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2$$

4. Solving for the final speed gives

$$v_f = \sqrt{v_i^2 + 2gy_i}$$

5. Calculations

$$v_i = 55\,mph = 55\left(\frac{mile}{hour}\right)\left(\frac{1609\ \ m}{1\,mile}\right)\left(\frac{1\,hour}{3600\,s}\right) = 24.6\,m/s$$

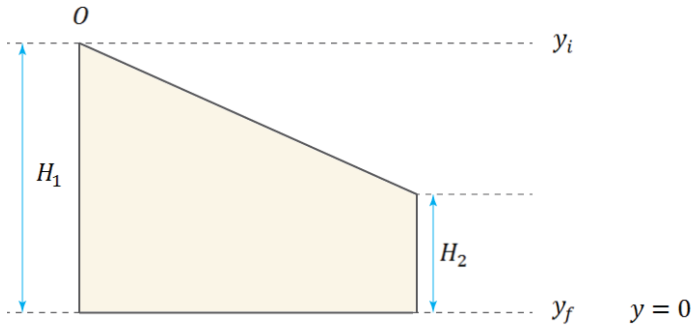
$$y_i = 90\,ft = 90\,ft\frac{0.348\,m}{1\,ft} = 27.4\,m$$

$$v_f = 33.8\ m/s = 75.6\,mph$$

6. We have got a proper dimension (speed in m/s). The numerical answer seems reasonable.

Example 7-3

A block is released from rest at the top of a frictionless ramp, at height H_1 (point O) above the base of the ramp. The ramp ends at height H_2 above the base, so that the block flies off and follows a two-dimensional projectile motion until it hits the ground. Find the speed of the block in terms of the given variables when it strikes the ground.



SOLUTION

1. Physics – motion with acceleration on incline, projectile motion, conservation of energy.

We could try to solve the problem in old-fashioned way. First we solve the “incline” problem, or motion with acceleration on the incline after using Newton’s second law to find the acceleration. Then we solve the projectile motion problem. However, it is not only a long way to go; it is not possible in this case since we are not given the horizontal size of the ramp.

Therefore we are going to use conservation of energy

2. The basic equation

$$K_i + U_i = K_f + U_f + f_k d$$

3. In this problem, there is only one force affecting the motion, namely gravity. We could split the motion into two phases, namely from the top of the ramp H_1 to the end at height H_2, and them the motion to the ground. But it is unnecessary. We may just consider the initial (on the top of the ramp) and final (on the ground) configurations, where $y_i = H_1,\ v_i = 0$ and $y_f = 0$. Then the equation

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

is greatly simplified

$$0 + mgH_1 = \frac{1}{2}mv_f^2 + 0$$

4. The speed of the block as it strikes the ground is

$$v_f = \sqrt{2gH_1}$$

5. No calculations (no data)

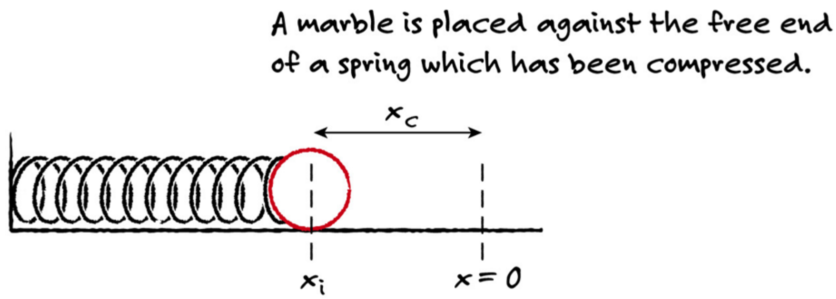
6. Let's check dimension for the speed $\sqrt{\frac{m}{s^2}m} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}$ correct, we got it as m/s.

The same speed is the same as dropping the block vertically from the same height.

Note: In the same way we may solve a roller coaster problem. If there is no resistance/friction, then the profile does not matter.

Example 7-4

A child's toy shoots a marble with a horizontal spring (spring constant k=11N/m). The marble has a mass of 16 g. If the spring is compressed 3.4 cm and then released, what is the speed of the marble when it leaves the spring? Assume that the marble experiences no resistive forces.



SOLUTION

1. Physics – spring, motion, conservation of energy.

2. The basic equation

$$K_i + U_i = K_f + U_f + f_k d$$

3. In this problem we have one only force, namely the spring force

$$\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2$$

Using the conditions $v_i = 0,\ x_f = 0$

$$0 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + 0$$

4. Solving the equation for the final speed gives

$$v_f = \sqrt{\frac{kx_i^2}{m}}$$

5. Calculations

$$v_f = \sqrt{\frac{(11\ N/m)(-0.034\ m)^2}{0.016\ kg}} = 0.89\ m/s$$

6. Check dimensions

[N]=[kg]*[m]/[s]^2 then for the speed

$$\sqrt{\frac{kg \cdot m}{s^2}\frac{1}{m}\frac{m^2}{kg}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}$$

correct!

The numerical answer seems reasonable to spring toys.

<u>Example 7-5</u>

As 18,000 kg *F/A-18E/F Super Hornet* lands on aircraft carrier *USS Harry S. Truman (CVN-75)*, its tail hook snags an arresting cable to slow it down. The cable stretches 80 m to stop the aircraft. Assume that the cable is attached to a spring with spring constant 10,000 N/m, and the coefficient of static friction between the aircraft's tires and the deck is 0.80.

What was the plane's landing speed?

SOLUTION

1. Physics – motion, spring force, friction, energy conservation

2. The basic equation $K_i + U_i = K_f + U_f + f_k d$

3. For the problem in hand (the initial kinetic energy went into spring potential energy and friction) we have

$$\frac{1}{2}mv_i^2 = \frac{1}{2}kx_f^2 + \mu mgx$$

4. Solving for v_i

$$v_i = \sqrt{\frac{kx_f^2}{m} + 2\mu gx}$$

5. Calculations

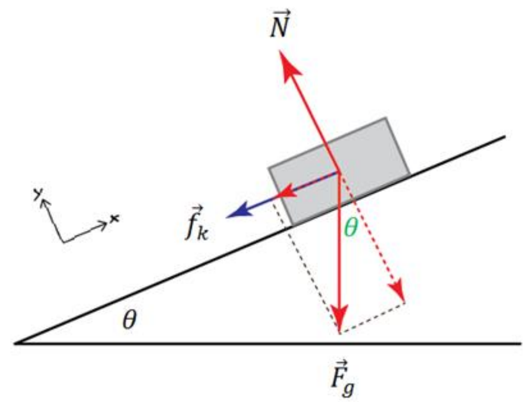
$$v_i = \sqrt{\frac{10000\ N/m\ \ (80\ m)^2}{18000\ kg} + 2 \cdot 0.8 \cdot 9.8\ \ m/s^2 \cdot 80\ m} = 69.3\ \ m/s$$

6. Let's see the landing speed in mph 69.3 $m/s = 155\ mph$. This is a very reasonable landing speed for F-18 Super Hornet.

Example 7-6

A 10,000-kg runaway truck with failed brakes is moving 70 mph just before the driver steers the truck up the runaway ramp with an inclination of 20^0. Assume that the coefficients of static and kinetic friction between the ramp and the truck are 0.95 and 0.8 respectively. A driver slammed on his brakes. What minimum length L must the ramp have to stop along it?

SOLUTION



1. Physics – three forces (gravity, friction, normal), motion along an incline and energy.

This problem can be solved by either the old way[4] (using Newton's laws + one dimensional kinematics) or using conservation of energy. Let's use conservation of energy.

2. The basic equation

$$K_i + U_i = K_f + U_f + f_k d$$

3. Working with conservation of energy we have to carefully choose the initial and final configurations and reference points for potentials. In this problem our choice is rather straightforward. We choose the initial point at the base of the ramp where $y_i = 0$, then

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f + f_\mu d$$

where d is the stopping distance.

The conditions $v_f = 0$, $y_f = d\sin\theta$, and the frictional force $f_\mu = \mu_s N = \mu_s mg\cos\theta$ (see chapter 5 for details). Note that we use the static coefficient assuming the brakes were not locked.

Then the conservation of energy can be written as

$$\frac{1}{2}mv_i^2 + 0 = 0 + mgd\sin\theta + \mu_s mgd\cos\theta$$

4. The equation above can be easily solved for d

$$d = \frac{v_i^2}{2g(\sin\theta + \mu_s\cos\theta)}$$

5. Calculations

$$70\ mph = 70\ \frac{mile}{h}\left(\frac{1609\ m}{1\ mile}\right)\left(\frac{1\ h}{3600\ s}\right) = 31\ m/s$$

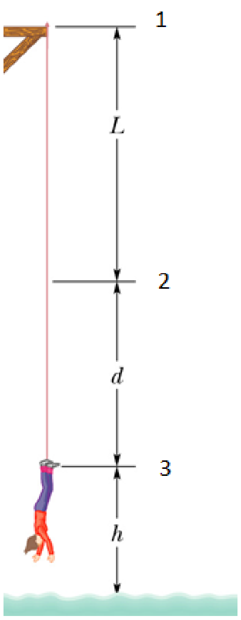
[4] see example 5-4 in chapter 5

$$d = \frac{(31\,m/s)^2}{2 \cdot 9.8\,m/s^2\,(\sin 20^0 + 0.95\cos 20^0)} = 40\,m$$

6. The dimension is correct and the numerical answer looks realistic. By the way the solution is identical to one in example 5-4.

Example 7-7

Student jumps off a bridge 52 m above a river with a bungee cord tied around his ankle. He falls 15 m before the bungee cord begins to stretch. Student's mass is 75 kg and the cord (spring) constant is k=50 N/m. If we neglect air resistance, estimate how far below the bridge the student would fall before coming to stop.



SOLUTION

1. Physics – gravity, spring force, motion, energy conservation

2. The basic equation $K_i + U_i = K_f + U_f + f_k d$

3. There are two forces involved, namely gravity and spring force. We have to carefully choose the reference points for our problem. For gravity we may use any reference point, but the reference point for the spring potential energy must be the spring equilibrium point (point 2 on the diagram).

Our initial starting point is (1), and the final point is (3). Sure, we may solve the problem in two steps (motion from 1 to 2, and then from 2 to 3), but we may eliminate intermediate steps.

Let's choose the first point (1) as a reference point for gravity $y_1 = 0$. The second point (point 2) is the reference point for the spring. The conditions are $v_i = 0$, $v_f = 0$. On the diagram $L = 15\,m$, and $L + d + h = 52\,m$ (we need to find $L + d$)

Then the general equation (without friction)

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}kx_f^2$$

can be written for $i = 1$ and $f = 3$ as

$$0 + 0 + 0 = 0 - mg(L + d) + \frac{1}{2}kd^2$$

4. In the quadratic equation above we are given L and k but we do not know d. Solving the quadratic equation would give the unknown d.

5. Calculations

The quadratic equation for d is

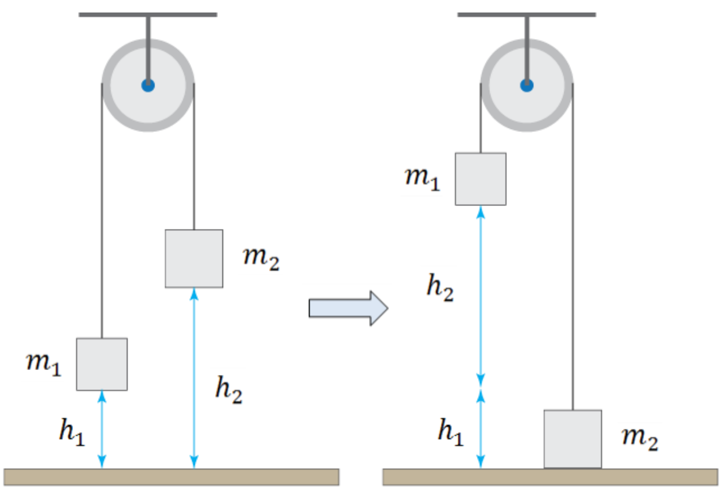
$$\frac{1}{2}50d^2 - 75 \cdot 9.8 \cdot d - 75 \cdot 9.8 \cdot 15 = 0 \quad or \quad 25d^2 - 735d - 11025 = 0$$

Solutions: $x_1 = 40\ m,\ x_2 = -11\ m$ We keep the first solution, since the second one corresponds to a point above (2). Then $L + d = 15\ m + 40\ m = 55\ m$. But the bridge is only 52 m above the river. The student will hit the water. (With this information and using conservation of energy we can even find his speed at the impact.)

6. With no experience in bungee jumping we could not say for sure if the numerical answer is reasonable. However we may check dimensions in the quadratic equation: for kx^2 it is N*m, for the other terms N*m too.

Example 7-8

Consider the Atwood machine. The two masses have the values m_1 and m_2. The system is released from rest with m_1 at height h_1 and m_2 at height h_2 from the floor. Use energy conservation to find the speed of m_2 just before it hits the floor.



SOLUTION

1. Physics – motion, energy conservation, gravity with TWO objects

2. The basic equation

$$K_i + U_i = K_f + U_f + f_k d$$

3. Since there are two objects we apply the conservation of energy for both of them. Keeping in mind that there is only gravitational potential energy in the problem we can write

$$K_{1i} + U_{1i} + K_{2i} + U_{2i} = K_{1f} + U_{1f} + K_{2f} + U_{2f}$$

Both the initial kinetic energies are zero, counting the gravitational potential energy from the floor, and since the masses have the same speed we get

$$m_1 g h_1 + m_2 g h_2 = \frac{m_1 v^2}{2} + \frac{m_2 v^2}{2} + m_1 g(h_1 + h_2) + 0$$

4. Solving for v

$$m_2gh_2 - m_1gh_2 = \frac{1}{2}(m_1 + m_2)\frac{v^2}{2}$$

$$v = \sqrt{\frac{2(m_2 - m_1)gh_2}{m_2 + m_1}}$$

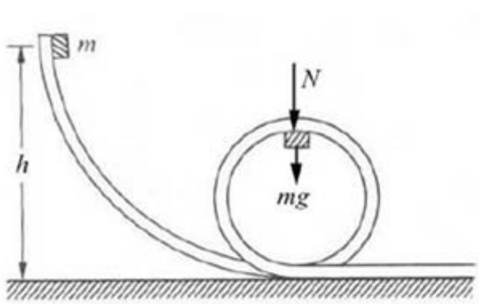
5. Calculations

There is nothing to calculate

6. Let's analyze the answer. The dimension is correct $[v] = \sqrt{[m^2]/[s^2]} = m/s$. In case of $m_1 = m_2$ we have $v = 0$ that looks right.

Example 7-9

An object of mass m is released from rest at a height h above the surface. The object slides along the inside of the loop-the-loop track consisting of a ramp and a circular loop of radius R shown in the figure. Assume that the track is frictionless. Calculate, in terms of the given quantities, the minimum release height h.



SOLUTION:

1. Physics. Energy, circular motion, forces.

2. The basic equations

$$K_i + U_i = K_f + U_f + f_k d$$

$$F_{net} = \frac{mv^2}{R}$$

3. This problem has two parts. First, we need to find the least speed the block has to have at the top of the loop to remain in contact with it there. Then we will look for the height h that would provide this speed. The solution to the first part can be found in example 5-8 (this lecture notes). Here are principal points. From the free-body diagram

$$-N - mg = m(-a) = m\left(-\frac{v^2}{R}\right)$$

Using conservation of energy gives

$$mgh = \frac{mv^2}{2} + 2mgR$$

4. If the block has the least speed to remain in contact, then it is on the verge of losing contact with the loop (falling away from the loop), which means $N = 0$. Thus

$$-mg = m\left(-\frac{v^2}{R}\right) \quad \text{and} \quad v^2 = gR$$

Using this v^2 in equation for conservation of energy

$$mgh = m\frac{gR}{2} + 2mgR$$

$$h = \frac{5}{2}R$$

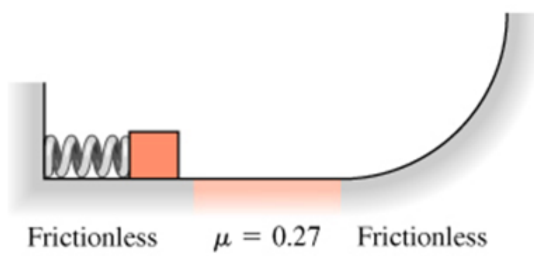
5. Calculations

There is nothing to calculate

6. Let's analyze the answer. The dimension is obviously right. The order of magnitude seems reasonable if one has experience with amusement parks.

<u>Example 7-10</u>

A 182 g block is launched by compressing a spring of constant k=200 N/m a distance of 15 cm. The spring is mounted horizontally, and the surface directly under it is frictionless. But beyond the equilibrium position of the spring end, the surface has coefficient of friction of μ=0.27. This frictional surface extends 85 cm followed by a frictionless curved rise, as shown in the figure



After launch, where does the block finally come to rest? You measure from the left end of the frictional zone.

SOLUTION

1. Physics: motion, spring energy, gravity, friction, conservation of energy

2. The basic equation (conservation of energy)

$$K_i + U_i = K_f + U_f + f_k d$$

3. After the spring is released the block travels back and forth losing kinetic and potential (spring or gravitational) energy to friction.

Eventually all initial spring potential energy is transferred to friction (heat). Since we are interested only in the final position of the block on the friction zone

$$\frac{1}{2}kx_i^2 = \mu mgd$$

Where d is the total distance travelled along the frictional zone.

4. Solving for the distance

$$d = \frac{1}{2}kx_i^2 \frac{1}{\mu mg} = \frac{kx_i^2}{2\mu mg}$$

5. Calculations

$$d = \frac{kx_i^2}{2\mu mg} = \frac{200\ N/m \cdot (0.15\ m)^2}{2 \cdot 0.27 \cdot 0.182\ kg \cdot 9.8\ m/s^2} = 4.67\ m$$

It is clear that the block travels more than once through the frictional part ($4.67/0.85 = 5.49$ or 5 full times, and on the six run the block stopped after travelling 0.49 part of the zone from the right end, or 0.51 part from the left end). Then the final position is $0.51 \cdot 0.85\ m = 0.43\ m$ from the left end.

6. Looking back. There is not much room for applying our common sense, but we may check the dimension

$$[d] = \frac{N \cdot m^2 \cdot s^2}{m \cdot kg \cdot m} = \frac{N \cdot s^2}{kg} = \frac{kg \cdot m}{s^2} \cdot \frac{s^2}{kg} = [m]$$

Correct.

8 Systems of particles

So far, we were dealing with essentially one particle problems. Our analysis was limited to motion with constant acceleration or we used conservation of energy when we were not interested in some details but cared about only initial and final states or configurations. Unfortunately, there are really very few problems which can be solved exactly by analysis.

On the basis of Newton's second law of motion, which gives the relation between the acceleration of any body and the force acting on it, any problem in mechanics can be numerically solved in principle. For example, if there are two bodies going around the sun, so that the total number of bodies is three, then analysis cannot produce a simple formula for the motion, and in practice the problem must be done numerically. That is the famous three-body problem, which so long challenged human powers of analysis; it is very interesting how long it took people to appreciate the fact that perhaps the powers of mathematical analysis were limited and it might be necessary to use the numerical methods. Today an enormous number of problems that cannot be done analytically are solved by numerical methods, and the old three-body problem in classical mechanics[5], which was supposed to be so difficult, is solved routinely on a personal computer.

However, there are also situations where both methods (analysis and numerical) fail: the simple problems we can do by analysis, and the moderately difficult problems by numerical, computational methods, but the very complicated problems we cannot do by either method. A complicated problem is, for example, the collision of two automobiles, or even the motion of the molecules of a gas. There

[5] The three-body problem in quantum mechanics is still one of the most challenging problems in physics despite using most powerful supercomputers.

are countless particles in a cubic millimeter of gas, and it would be ridiculous to try to make calculations with so many variables (about 10^{17} - a hundred million billion). Anything like the motion of the molecules or atoms of a gas or a block or iron, or the motion of the stars in a globular cluster, instead of just two or three planets going around the sun—such problems we cannot do directly, so we have to seek other means.

In the situations in which we cannot follow details, we need to know some general properties, that is, general theorems or principles which are consequences of Newton's laws. One of these is the principle of conservation of energy, which was discussed in Chapter 7. Another is the principle of conservation of momentum, the subject of this chapter.

Another reason for studying mechanics further is that there are certain patterns of motion that are repeated in many different circumstances, so it is good to study these patterns in one particular circumstance. For example, we shall study collisions; different kinds of collisions have much in common. In the flow of fluids, it does not make much difference what the fluid is, the laws of the flow are similar.

In our discussion of Newton's laws, it was explained that these laws are a kind of program that says "Pay attention to the forces," and that Newton told us only two things about the nature of forces. In the case of gravitation, he gave us the complete law of the force. In the case of the very complicated forces between atoms, he was not aware of the right laws for the forces; however, he discovered one rule, one general property of forces, which is expressed in his Third Law, and that is the total knowledge that Newton had about the nature of forces—the law of gravitation and this principle, but no other details.

This principle is that action equals reaction. What is meant is something of this kind: Suppose we have two small bodies, say particles, and suppose that the first one exerts a force on the second one, pushing it with a certain force. Then, simultaneously, according to Newton's Third Law, the second particle will push on the first with an equal force, in the opposite direction; furthermore, these forces effectively act in the same line. This is the hypothesis, or law, that Newton proposed, and it seems to be quite accurate, though not exact (we shall discuss the errors later). For the moment we shall take it to be true that action equals reaction. Of course, if there is a third particle, not on the same line as the other two, the law does not mean that the total force on the first one is equal to the total force on the second, since the third particle, for instance, exerts its own push on each of the other two. The result is that the total effect on the first two is in some other direction, and the forces on the first two particles are, in general, neither equal nor opposite. However, the forces on each particle can be resolved into parts, there being one contribution or part due to each other interacting particle. Then each pair of particles has corresponding components of mutual interaction that are equal in magnitude and opposite in direction.

8.1 Momentum

Momentum is a word that has multiple meanings in everyday language, but only a single meaning in physics. The *linear momentum* of a particle is a vector $\vec{p}$, defined as

$$\vec{p} = m\vec{v} \tag{8.1}$$

in which m is the mass of the particle and $\vec{v}$ is its velocity. (The adjective *linear* is often dropped, but it serves to distinguish $\vec{p}$ from *angular momentum*, which will be introduced later). Since m is always a positive quantity, then $\vec{p}$ and $\vec{v}$ have the same direction. The SI unit for momentum is *kg·m/s,* with no special name for this combination of units.

Newton actually expressed his second law of motion in terms of momentum

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} \tag{8.2}$$

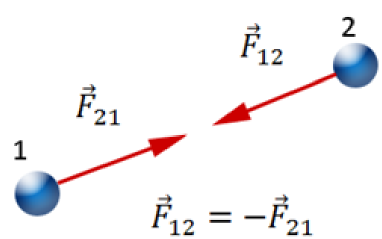
it is very straightforward

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Thus we have two equivalent expressions of Newton's second law.

8.2 The linear momentum of a system of particles

Suppose, for simplicity, that we have just two interacting particles, possibly of different mass, and numbered 1 and 2. The forces between them are equal and opposite; what are the consequences?



According to Newton's Second Law, force is the rate of change of the momentum with respect to time, so

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{21} = -\vec{F}_{12} = -\frac{d\vec{p}_2}{dt}$$

then it follows that

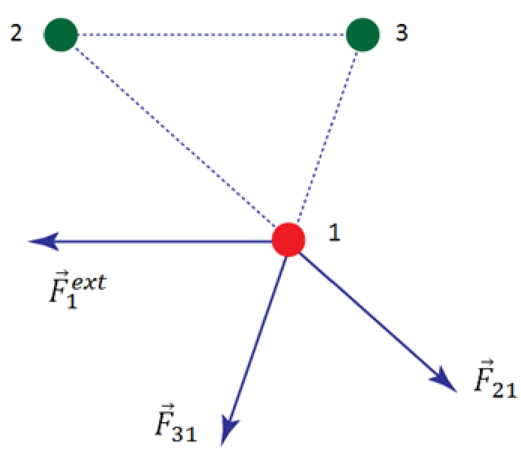
$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2)$$

There is assumed to be no other force in the problem. If the rate of change of this sum is always zero, that is just another way of saying that the quantity $(\vec{p}_1 + \vec{p}_2)$ or $m\vec{v}_1 + m\vec{v}_2$ does not change. We have now obtained the result that the total momentum of the two particles does not change because of any mutual interactions between them. This statement expresses the law of conservation of momentum in that particular example. We conclude that if there is any kind of force, no matter how complicated, between two particles, and we measure or calculate $m\vec{v}_1 + m\vec{v}_2$, that is, the sum of the two momenta, both before and after the forces act, the results should be equal, i.e., the total momentum is a constant.

If we extend the argument to three or more interacting particles in more complicated circumstances, it is evident that so far as internal forces are concerned, the total momentum of all the particles stays constant, since an increase in momentum of one, due to another, is exactly compensated by the decrease of the second, due to the first. That is, all the internal forces will balance out, and therefore cannot change the total momentum of the particles. If there are no forces from the outside (external forces), there are no forces that can change the total momentum; hence the total momentum is a constant.

It is worth describing what happens if there are forces that do not come from the mutual actions of the particles in question: suppose we isolate the interacting particles. If there are only mutual forces, then, as before, the total momentum of the particles does not change, no matter how complicated the forces. On the other hand, suppose there are also forces coming from the particles outside the isolated group. Any force exerted by outside bodies on inside bodies, we call an external force.

Now consider a system of three particles[6], each with its own mass, velocity and linear momentum. The particles may interact with each other, and external forces may act on them as well.



The system as a whole has a total linear momentum $\vec{P}$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$$

If we take the time derivative, we find that

$$\frac{d\vec{P}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt}$$

[6] A generalization to a system of n particles is straightforward, and you may consider it as a good exercise

For every particle

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i^{ext} + \sum_{j\neq i}^{3} \vec{F}_{ji}$$

Then

$$\frac{d\vec{P}}{dt} = \vec{F}_1^{ext} + \vec{F}_{21} + \vec{F}_{31} + \vec{F}_2^{ext} + \vec{F}_{12} + \vec{F}_{32} + \vec{F}_3^{ext} + \vec{F}_{13} + \vec{F}_{23}$$

According to Newton's third law (action-reaction)

$$\vec{F}_{21} = -\vec{F}_{12}, \quad \vec{F}_{31} = -\vec{F}_{13}, \quad \vec{F}_{32} = -\vec{F}_{23}$$

and

$$\frac{d\vec{P}}{dt} = \vec{F}_1^{ext} + \vec{F}_2^{ext} + \vec{F}_3^{ext} = \vec{F}^{ext}$$

where $\vec{F}^{ext}$ is the net external force, and if $\vec{F}^{ext} = 0$ then

$$\frac{d\vec{P}}{dt} = 0 \quad \Rightarrow \vec{P} = const \tag{8.3}$$

momentum of the entire system of particles is conserved.

Principle of Conservation of Linear Momentum

If the net force $\vec{F}^{ext}$ on an N-particle system is zero, the system's total mechanical momentum

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_N\vec{v}_N \tag{8.4}$$

is constant

This is one of the most important results in classical (Newtonian) mechanics.

Note that momentum of a single particle is not necessarily conserved, only the total momentum of a closed system.

8.3 Newton's second law for a system of particles

For a system of point-like particles (see above).

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext} \tag{8.5}$$

At this point we introduce *the center of mass* for a system of *n* particles

$$\vec{R}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i \tag{8.6}$$

where

$$M = m_1 + m_2 + \cdots m_n = \sum_{i=1}^{n} m_i \tag{8.7}$$

is the total mass of the system.

If the particles are distributed in three dimensions, the center of mas can be identified by three coordinates

$$x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \qquad y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \qquad z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i. \tag{8.8}$$

An ordinary object contains so many particles (atoms) that we can treat it as a continuous distribution of matter. Then the sums above become integrals.

In this chapter we deal with particle-like systems or simple enough symmetrical objects when the position of the center of mass is known, e.g. the center of mass of a *uniform rod of length L* is at L/2 the center of mass of a *uniform spherical ball* is at the center of the ball.

Differentiating the definition for the center of mass with respect to time gives

$$\vec{v}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i \tag{8.9}$$

where $\vec{v}_i$ is the velocity of the $i -$ th particle and $\vec{v}_{CM}$ is the velocity of the center of mass.

Differentiating the last equation with respect to time gives

$$\vec{a}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{a}_i = \frac{1}{M}\sum_{i=1}^{n} \vec{F}_i = \frac{1}{M}\vec{F}^{ext}.$$

or

$$Ma_{CM} = \vec{F}^{ext} \tag{8.10}$$

Thus the center of mass of a system moves as though all of the mass were concentrated there and all external forces were applied there.

There are two obvious results from this equation.

- If the net force acting on a system is zero, then a system moves with constant velocity
  $\vec{v}_{CM} = contant$ for $\vec{F}^{ext} = 0$
- And if the center of mass of a system was not moving $\vec{v}_{CM} = 0$, then the position of the center of mass $\vec{R}_{CM}$ does not change despite individual positions of particles may change.
  $\vec{R}_{CM} = constant$ for $\vec{v}_{CM} = 0$ (if $\vec{F}^{ext} = 0$)

8.4 Impulse and Linear Momentum

Using Newton's second law in momentum form

$$\vec{F} = \frac{d\vec{p}}{dt}$$

we can write

$$d\vec{p} = \vec{F}(t)dt$$

in which $\vec{F}(t)$ is a time-varying force. Let's integrate this equation over the time interval Δt from an initial time t_i to a final time t_f. We obtain

$$\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

The left side of the equation is $\vec{p}_f - \vec{p}_i$ the change in linear momentum of an object. The right side, which measures both the strength and the duration of the force, is called the *impulse* $\vec{J}$.

$$\vec{J} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{8.11}$$

The last equation can be written in component form as

$$p_{fx} - p_{ix} = J_x = \int_{t_i}^{t_f} F_x(t)dt, \quad p_{fy} - p_{iy} = J_y = \int_{t_i}^{t_f} F_y(t)dt, \quad p_{fz} - p_{iz} = J_z = \int_{t_i}^{t_f} F_z(t)dt.$$

If F_{avg} is the average magnitude of the force, we can write the magnitude of the impulse for every component as

$$J = p_f - p_i = F_{avg}\Delta t. \tag{8.12}$$

where Δt is the duration of the action of the force.

The SI units for impulse are [N·s]=[kg·m/s], which are the same units as momentum.

8.5 Collisions

A collision is an isolated event in which two or more bodies exert relatively strong forces on each other for a relatively short period of time.

- "relatively strong forces" – comparing to external forces
- "relatively short period of time" – so that external force produces negligible impulse (change in momentum)

Let's consider a system of two colliding bodies. If there to be a collision, then one of the bodies must be moving, so that the system has a certain kinetic energy and a certain linear momentum before the collision. During the collision, the kinetic energy and linear momentum of each body are changed by the force from the other body. In this section the discussion is limited to collisions in system that are *closed* (no mass enters or leaves them) and *isolated* (no net external forces act on the bodies within the system).

Kinetic energy and two types of collisions

- **Elastic collisions** – if the total kinetic energy of the system is unchanged by the collisions (the kinetic energy of the system is conserved).
  This type of collisions is very unlikely in everyday life[7], but more common for collisions of atomic or nuclear particles.
- **Inelastic collisions** – where the kinetic energy of the system is not conserved.
  There is a special kind of inelastic collisions called *completely inelastic collision* if two bodies stick together after the collision and move as a single object.

Regardless of the forces acting between bodies during the collision, and regardless to what happens to the total kinetic energy of the system, the total linear momentum is always conserved. That is quite interesting, namely we can relate momenta before and after a collision even if we do not know details of the forces during the collision.

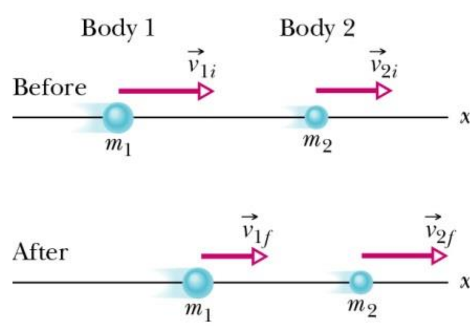
In a closed isolated system, the linear momentum of each colliding body may change but the total linear momentum $\vec{P} = m\vec{v}_1 + m\vec{v}_2$ of the system cannot change, whether the collision is elastic or inelastic.

8.5.1 Inelastic collision in one dimension

We can write the law of conservation of linear momentum for a two-body system as

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

Figure shows two bodies just before and just after they have a one-dimensional collision.



[7] In some situations we can approximate a collision of common bodies as elastic - for example a dropping a Superball (also known as a bouncy ball) on a hard floor.

Because the motion is one-dimensional, we can drop the overall arrows for vectors and use only components along the axis. Then, using $p = mv$ we can write

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{8.13}$$

Since there is only one equation, we may have only one unknown.

Now let's consider a *completely inelastic collision* (after the collision the bodies stick together moving with the same velocity)

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

then the final velocity of the combined body

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \tag{8.14}$$

The last equation last equation can be analyzed for various masses m_1 and m_2. We will do that for $v_{2i} = 0$ (or the body with mass m_2 happens to be initially at rest)

$$v_f = \frac{m_1}{m_1 + m_2} v_{1i} \tag{8.15}$$

We note that the combined system moves in the same direction as initially moving body.

Let's consider three combinations of masses

- **A massive target**: $m_1 \ll m_2$ then $v_f \approx 0$ i.e., the combined system is moving with very small velocity.
- **A massive projectile**: $m_1 \gg m_2$ then $v_f \approx v_{1i}$ i.e., the combined system is moving with velocity very close to the velocity of the initially moving body.
- **Equal masses**: $m_1 = m_2$ then $v_f = v_{1i}/2$ i.e., the combined system moves with the final velocity that is half of the initial velocity.

8.5.2 Elastic collision in one dimension

In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.

$$\begin{aligned} &m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \qquad conservation\ of\ linera\ momentum \\ &\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad conservation\ of\ kinetic\ energy \end{aligned} \tag{8.16}$$

These two equations describe elastic collision of two bodies in one dimension. Having two equations we can solve the system for two unknowns. If we know the masses and initial velocities, then we can solve these equations for the velocities after the collision v_{1f} and v_{2f}.We rewrite the momentum equations as

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i}) \tag{8.17}$$

and the kinetic energy equation can be rewritten as

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

noting that $(a^2 - b^2) = (a - b)(a + b)$ we rewrite the last equation

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \tag{8.18}$$

Dividing the left and the right sides of the last equation (8.18) by the corresponding sides of the linear momentum equation (8.17) gives

$$v_{1i} + v_{1f} = v_{2f} + v_{2i} \quad or \quad v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

The results tells us that for one dimensional elastic head-on collisions, the relative speed of the two objects after the collisions has the same magnitude as before (but opposite direction), no matter what the masses are.

From the last equation,

$$v_{2f} = v_{1i} - v_{2i} + v_{1f}$$

Substituting this result into (8.17) gives

$$m_1(v_{1i} - v_{1f}) = m_2(v_{1i} - v_{2i} + v_{1f} - v_{2i}) \;\; or \;\; (m_1 - m_2)v_{1i} + 2m_2 v_{2i} = (m_1 + m_2)v_{1f}$$

Solving for v_{1f}

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \tag{8.19}$$

Having v_{1f} we can easily (simply algebra) find v_{2f} from (8.17).

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \tag{8.20}$$

These equations express the final velocities in terms of initial velocities.

Let us look at a few special situations. For clarity of our analysis we assume that a projectile body m_1 moves toward a target body m_2 that is initially at rest $v_{2i} = 0$. Then

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \tag{8.21}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} \tag{8.22}$$

- **Equal masses**: If $m_1 = m_2$ then

  $$v_{1f} = 0 \;\; and \;\; v_{2f} = v_{1i}$$

  which we may call a pool player result. It predicts that after a head-on collision of bodies with equal masses, body 1 (initially moving) stops dead and body 2 (initially at rest) takes off with the initial speed of body 1.

- **A massive target:** A massive target means that $m_2 \gg m_1$ (for example firing a golf ball into a cannonball. Then

$$v_{1f} \approx -v_{1i} \quad and \quad v_{2f} \approx \frac{2m_1}{m_2} v_{2i}$$

This tells us that body 1 (the golf ball) simply bounces back along its incoming path, its speed is essentially unchanged. Body 2 (cannonball) moves forward at low speed. At this is what we should expect

- **A massive projectile:** This is the opposite case; that is $m_1 \gg m_2$. This time we fire a cannonball into a golf ball.

$$v_{1f} \approx v_{1i} \quad and\ v_{2f} \approx 2v_{1i}$$

This tells us that body 1 (the cannonball) simply keeps on going, scarcely slowed by the collision. Body 2 (the golf ball) charges ahead at twice the speed of the cannonball.

8.5.3 Collisions in two dimensions

Conservation of momentum and energy can also be applied to collisions in two and three dimensions, and the vector nature of momentum is especially important.

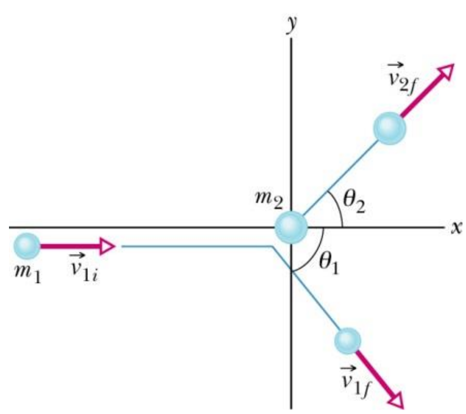
For two-dimensional collisions in a closed isolated system, the total linear momentum still be conserved

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

If the collision is also elastic (a special case), then the total kinetic energy is also conserved

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}.$$

For example, Figure shows a glancing collision (it is not head-on) between a projectile body and target body initially at rest.



It is more useful if we write conservation of momentum in term of components on an xy coordinate system.

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2 \qquad x - component$$
$$0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2 \qquad y - component$$

We can also write for elastic collisions

$$\frac{1}{2}m_1v_{1i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

These three equations have seven variables: two masses, three speeds and two angles. If we know any four of these quantities, we can solve the three equations for the remaining three quantities.

8.6 The center of mass of solid bodies*

Calculating the center of mass for a solid object is rather more complicated exercise than finding the center of mass of a system of point-like particles. However, we apply the same basic idea thinking about a solid object as a system containing a large number of particles with continuous mass distribution. Let's divide the object into elements of mass Δm_i with coordinates x_i , y_i , $z_{i.}$ Then, the x coordinate of the center of mass is approximately

$$x_{CM} \approx \frac{\sum_i x_i \Delta m_i}{M}$$

Using calculus we can write exactly

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M}\int xdm \tag{8.23}$$

where dm is the element of mass and the integral is taken over the extent of the object. The total mass is given simply by

$$M = \int dm$$

Likewise we can define y_{CM} and z_{CM}.

For a solid three-dimensional body the element of mass is just $dm = \rho dV$, where dV is the element of volume, and ρ is the density (that can be variable, namely $\rho(x, y, z)$. For a laminar body (i.e. a uniform sheet of material) the element of mass is $dm = \sigma dA$, where $\sigma(x, y)$ is the mass per unit area of the body, and dA is an area element. Finally, for a body in the form of a thin wire we have $dm = \lambda\, ds$, where λ is the mass per unit length and ds is an element of arc length along the wire.

Let's find the center of mass of a rod of length L that has a uniform density λ. The total mass of the rod is

$$M = \int_0^L \lambda dx = \lambda L$$

and the center of mass is located at

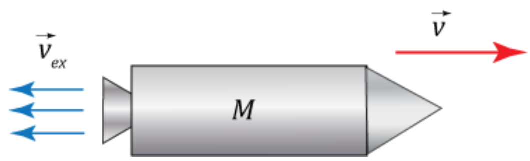
$$x_{CM} = \frac{1}{M}\int_0^L x\lambda dx = \frac{1}{\lambda L}\frac{\lambda L^2}{2} = \frac{L}{2}.$$

Two- and three- dimensional cases require evaluation of multiple integrals. Such integration goes beyond this course.

8.7 Dynamics of Bodies of Variable Mass; Rocket propulsion

Conservation of linear momentum is the foundation for the motion in space (where there is no any gas atmosphere).

Consider a rocket with mass M travelling in the positive x direction with speed v. Instead of using notation v_x for components we will write just v. The thrust of the rocket engine is created by ejecting the fuel combustion products in a direction opposite to the motion of the rocket with the exhaust speed v_{ex} relative to the rocket. Since the rocket is ejecting mass, its mass M is decreasing.



At time t, the linear momentum is mv, while at time $t + dt$ the rocket mass is $(m + dm)$, where dm is negative, and its momentum is $(m + dm)(v + dv)$. The ejected fuel at that time has mass $-dm$ and velocity $(v - v_{ex})$ relative to the ground. Then the total momentum at time $t + dt$ is

$$p(t + dt) = (m + dm)(v + dv) - dm(v - v_{ex}) = mv + mdv + dm\, v_{ex}$$

where we neglected the small term $dm\, dv$ (an infinitesimal of the second order). Thus, the change in the total momentum is

$$dp = p(t + dt) - p(t) = m\, dv + dm\, v_{ex}$$

If there is an external force F_{ext} acting on the rocket, then the change of momentum is $F_{ext}dt$. Without an external force the linear momentum is conserved and $dp = 0$. Therefore

$$m\, dv = -dm\, v_{ex} \tag{8.24}$$

or with dime derivatives we get

$$m\frac{dv}{dt} = -v_{ex}\frac{dm}{dt} \tag{8.25}$$

where dm/dt is the rate at which the rocket is ejecting mass. This equation describes the motion of rockets in the absence of external forces.

Equation (8.25) looks just like Newton's second law where the term on the right is called the thrust:

$$F_{thrust} = -v_{ex}\frac{dm}{dt}$$

This force is positive since dm/dt is negative.

Equation (8.24) can be solved using separation of variables (we assume that the exhaust speed is constant)

$$dv = -v_{ex}\frac{dm}{m}$$

Assuming that at $t = 0$ the mass of the rocket was m_0 and the initial velocity v_0 and integrating both sides

$$\int_{v_0}^{v} dv = -v_{ex} \int_{M_0}^{M} \frac{dm}{m}$$

we easily get

$$v - v_0 = v_{ex} \ln \frac{m_0}{m}$$

or

$$v(t) = v_0 + v_{ex} \ln \frac{m_0}{m}$$

where m rocket's mass at any time. Thus equation describes the change in the velocity of the rocket when its mass changes from m_0 to m. If the engine is applied to slow down the motion then we replace v_{ex} on $-v_{ex}$.

Let us quickly evaluate efficiency of delivering a payload to a low orbit around Earth. Assuming the initial speed $v_0 = 0\ m/s$, the final speed 8 km/s, the exhaust speed 4 km/s, and neglecting the force of gravity (too optimistic assumption) we get for the original mass m_0 to be about 90% of fuel.

A crude estimation for flying to Mars and come back gives for a payload about 1/1500 of the starting mass.

8.8 Examples

Example 8-1

A marksman holds a rifle of mass $m_R = 3.0\ kg$ loosely in his hands, so as to let it recoil freely when fired. He fires a bullet of mass $m_B = 5.0\ g$ horizontally with a velocity relative to the ground of $v_B = 300\ \ m/s$. What is the recoil velocity of the rifle? What is the kinetic energy of the bullet? Of the rifle?

SOLUTION:

1. Physics – conservation of linear momentum for the closed isolated system. We may assume the net external force from the marksman as zero while the bullet moves inside the barrel.

2. The basic equation

$$m_1\vec{v}_1 + m_2\vec{v}_2 = constant$$

or

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

3. Initially the rifle and bullet were not moving $v_{Bi} = 0,\ \ v_{Ri} = 0$ then for our one-dimensional case

$$0 = m_B v_{Bf} + m_R v_{Rf}$$

4. Solving for v_{Rf}

$$v_{Rf} = -\frac{m_B}{m_R} v_{Bf}$$

For kinetic energies (we only need the definition, there is no conservation of energy)

$$K_B = \frac{1}{2} m_B v_{Bf}^2 \qquad K_R = \frac{1}{2} m_R v_{Rf}^2$$

5. Calculations

$$v_{Rf} = -\frac{0.005\ kg}{3.0\ kg} 300\ m/s = -0.500\ m/s$$

$$K_B = \frac{1}{2} 0.005\ kg \cdot (300\ m/s)^2 = 225\ J$$

$$K_R = \frac{1}{2} 3.0\ kg \cdot (-0.5\ m/s)^2 = 0.375\ J$$

6. Looking back.

Units are correct. The recoil speed looks as a true one. It is interesting to note that while the bullet and rifle have the same magnitude of the final momenta; their kinetic energies are quite different because of large difference in speeds.

To satisfy our curiosity, let us derive the ratio of kinetic energy of the rifle to bullet

$$\frac{K_R}{K_B} = \frac{\frac{1}{2} m_R v_{Rf}^2}{\frac{1}{2} m_B v_{Bf}^2} = \frac{m_R \left(\frac{m_B}{m_R} v_{Bf}\right)^2}{m_B v_{Bf}^2} = \frac{m_B}{m_R}$$

Thus the heavier the rifle, the less kinetic energy is transferred to it.

Example 8-2

A heavy wooden crate rests on a floor. A bullet is fired horizontally into the crate so that the bullet stopping in it. How far will the block slide before coming to a stop? The mass of the bullet is 16.0 g, the mass of the block is 70.0 kg, the bullet's impact speed is 300 m/s and the coefficient of kinetic friction between the crate and the wooden floor is 0.22. (Assume that the bullet does not cause the crate to spin.)

SOLUTION:

1. Physics – a) conservation of linear momentum for the closed isolated system (assuming we may disregard any external forces while the bullet penetrates the crate), b) conservation of energy after the collision).

2. The basic equation:

for the collision (note that the energy is NOT conserved during the inelastic collision)

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

and after the collision

$$\frac{1}{2}(m_1 + m_2)v_f^2 = \mu_k (m_1 + m_2) g x$$

3. Initially the crate was not moving $v_{2i} = 0$, but then the system "bullet + crate" moves as a whole

$$m_1 v_{1i} = (m_1 + m_2) v_f$$

and for the stopping x distance we apply conservation of energy

$$\frac{1}{2}(m_1 + m_2)v_f^2 = \mu_k (m_1 + m_2) g x$$

4. Solving first for v_f

$$v_f = \frac{m_1}{m_1 + m_2} v_{1i}$$

then using conservation of energy gives

$$x = \frac{v_f^2}{2\mu_k g} = \frac{1}{2\mu_k g}\left(\frac{m_1}{m_1 + m_2}\right)^2 v_{1i}^2$$

5. Calculations

$$x = 0.00109\ m\ \approx 0.1\ cm$$

6. Looking back.

The dimension of x is correct (meters). The stopping distance is small (less than half an inch) but reasonable. It is interesting to note that Hollywood movies are so wrong when an object (a body) flies far away with high speed after being hit by a bullet. (See Mythbusters, season 2005, episode 25. Myths tested: Can a person be blown away by a bullet?)

Example 8-3

You have been called to testify in a trial involving a head-on collision. Car A weighs 1500 lb and was traveling eastward with an initial speed of v_A. Car B weighs 1100 lb and was traveling westward at an initial speed of $v_B = 45\ mph$. The cars locked bumpers and slid eastward with their wheels locked for 19 ft before stopping. The coefficient of kinetic friction between the tires and the road was measured to be $\mu_k = 0.75$.

How fast (in mph) was car A travelling just before the collision? (note – English units are used in U.S. legal proceedings.)

SOLUTION:

1. Physics – completely inelastic collision + dissipation of kinetic energy through friction (conservation of energy). (This example is somewhat similar to example 9-2.)

2. The basic equations

$$m_A v_A + m_B v_B = (m_A + m_B)v_f \qquad \text{completely inelastic collision}$$

$$\frac{1}{2}(m_A + m_B)v_f^2 = \mu_k (m_A + m_B)gx \quad \text{conservation of energy (after)}$$

3. Assume that car A was moving in the positive x-direction (eastward), and then the direction of card B was negative (westward) (keeping correct signs is important!)

$$m_A v_A - m_B v_B = (m_A + m_B)v_f$$

4. From conservation of energy we can easily find the final speed of the wreckage

$$v_f = \sqrt{2\mu_k gx}$$

Using it in the momentum equation

$$v_A = \frac{m_B}{m_A} v_B + \frac{m_A + m_B}{m_A}\sqrt{2\mu_k gx}$$

5. Calculations

First we need to switch to SI units for speed and distance, but we can keep English units for masses since they are entering equation as ratios.

$$45\ mph = 45\ \frac{mile}{h}\left(\frac{1609\ m}{1\ mile}\right)\left(\frac{1\ h}{3600\ s}\right) = 20\ m/s$$

$$19\ ft = 19\ ft\ \frac{1.0\ m}{3.28\ ft} = 5.8\ m$$

$$v_A = \frac{1100\ lb}{1500\ lb} 20\ m/s + \frac{1100\ lb + 1500 lb}{1500\ lb}\sqrt{2 \cdot 0.75 \cdot 9.8\ m/s^2 \cdot 5.8\ m} = 30.7\ m/s,$$

$$30.7\ m/s = 30.7 \frac{m}{s}\left(\frac{1\ mile}{1609\ m}\right)\left(\frac{3600\ s}{1\ h}\right) = 69\ mph$$

6. Looking back.

Units are correct. If the wreckage slides eastward, and the masses or the cars are comparable, then car A must have been traveling faster than car B when they collided. It corresponds to our result.

Example 8-4

It is well known that bullets and other missiles fired at Superman simply bounce off his chest. Suppose that a gangster sprays Superman's chest with 5.0 g bullets at a rate of 100 bullets/min, and the speed of each bullet is 500 m/s. Suppose too that the bullets rebound straight back with no change speed. What is the magnitude of the average force on Superman's chest from the stream of bullets?

SOLUTION:

1. Physics –force, momentum.

2. The basic equation

The force can be found from the impulse (change of momentum)

$$J = \vec{p}_f - \vec{p}_i = F_{avg}\Delta t.$$

3. Assume that bullets were incoming from positive x direction (or negative initial velocity)

$$mv_0 + mv_0 = F_{avg}\Delta t.$$

or

$$2mv_0 = F_{avg}\Delta t$$

4. Solving the last equation for F_{avg}

$$F_{avg} = \frac{2mv_0}{\Delta t}$$

5. Calculations

$$F_{avg} = \frac{2 \cdot 0.005\ kg\ \cdot 100 \cdot 500\ m/s}{60\ s} = 8.3\ N$$

6. Looking back.

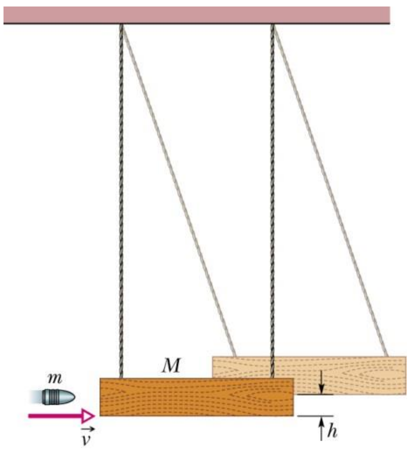
Units are correct. Commons sense cannot be applied to Superman, but the force (8.3 N or about 2 lb) looks reasonable and small (unlike in movies!)

<u>Example 8-5</u>

The ballistic pendulum was used to measure the speeds of bullets before electronic timing devices were developed. The version is shown in Figure consists of a large block of wood of mass M=5.4 kg, hanging from two long cords. A bullet of mass 9.5 g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance h=6.3 cm before the pendulum comes momentarily to rest at the end of its arc.

a) What is the speed of the bullet just prior to the collision?
b) Find the fraction of kinetic energy lost in the ballistic-pendulum collision.

SOLUTION:



1. Physics – completely inelastic collision + conservation of mechanical energy.

2. The basic equations

$$mv_i + Mv_{Bi} = (m+M)v_f$$

$$\frac{1}{2}(m+M)v_f^2 = (m+M)gh$$

3. Initially the wooden block was not moving $v_{Bi} = 0$

$$mv_i = (m+M)v_f$$

4. Solving for v_f

$$v_f = \frac{m}{m+M}v_i$$

Using it in the energy equation

$$\frac{1}{2}v_f^2 = gh \quad or \quad \frac{1}{2}\frac{m^2}{(m+M)^2}v_i^2 = gh$$

then

$$v_i = \frac{m+M}{m}\sqrt{2gh}$$

Fraction of kinetic energy lost in the collision

$$R = \frac{K_i - K_f}{K_i} = 1 - \frac{\frac{1}{2}(m+M)v_f^2}{\frac{1}{2}mv_i^2} = 1 - \frac{m+M}{m}\frac{m^2}{(m+M)^2}\frac{v_i^2}{v_i^2} = 1 - \frac{m}{m+M} = \frac{M}{m+M}$$

5. Calculations

$$v_i = \frac{(0.0095\ kg + 5.4\ kg)}{0.0095\ kg}\sqrt{2\cdot 9.8\ m/s^2\cdot 0.063m} = 630\ m/s\,, \qquad R = \frac{5.4\ kg}{0.0095\ kg + 5.4\ kg} = 0.998$$

6. Looking back.

Units are correct. The speed of the bullet matches common speeds for rifles.

Example 8-6

A careless physics professor is in the path of a pack of stampeding elephants when Tarzan swings in to the rescue on a rope vine, hauling him off to safety. The length of the vine is 16 m, and Tarzan starts his swing with the rope horizontal. If the professor's mass is 80 kg and Tarzan's mass is also 80 kg to what height above the ground will the pair swing? (Assume the rope is vertical when Tarzan grabs the professor.)

SOLUTION:

1. Physics – completely inelastic collision + conservation of mechanical energy.

2. The basic equations

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

$$\frac{1}{2} M v^2 = Mgh$$

3. There are three phases of motion in the problem, namely a) Tarzan swings from the tree (conservation of mechanical energy), b) picking up the professor (completely inelastic collision), and c) the pair swings up (conservation of mechanical energy). Assuming that 1 stands for Tarzan and 2 for the professor one can write

$$m_1 g R = \frac{1}{2} m_1 v_1^2 \qquad phase\ 1$$

$$m_1 v_1 = (m_1 + m_2) v_f \qquad phase\ 2$$

$$(m_1 + m_2)\frac{v_f^2}{2} = (m_1 + m_2) g h \qquad phase\ 3$$

where R is the length of the vine, and h is the height above the ground for Tarzan and the professor.

4. Solving for h (after simple algebra)

$$h = \left(\frac{m_1}{m_1 + m_2}\right)^2 R$$

5. Calculations

$$R = \left(\frac{80\ kg}{80\ kg + 80\ kg}\right)^2 16\ m = 4\ m$$

6. Looking back.

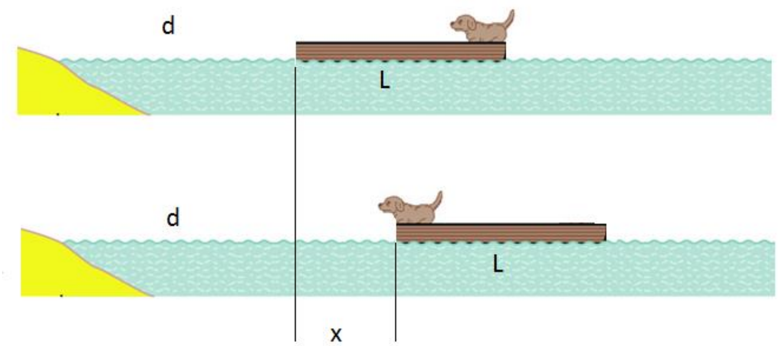
Units are correct. it is interesting that the final height is one quarter of the rope, not one half as one would expect using just conservation of energy.

<u>Example 8-7</u>

A 4.5 kg dog stands on an 18 kg flatboat $L + d = 6.1\ m$ from the shore. The dog walks $L = 2.4\ m$ along the boat toward the shore, and then stops. Assuming there is no friction between the boat and the water, find how far the dog is then from the shore.

Hint: use a stationary point as a reference point +having a diagram with proper notations helps a lot.

Note that the center of mass for a uniform boat is in the middle (at L/2).



SOLUTION:

1. Physics: a system of particles (the dog and boat)

2. The basic equation

$$Ma_{CM} = \vec{F}^{ext}$$

3. Let 1 be the dog and 2 be the boat. We choose the origin at the shore line.

The net force acting on the system is zero (gravity is balanced by the buoyant force) $\vec{F}^{ext} = 0$. Initially the system was stationary $v_{CM} = 0$, then the position of the center of mass does not change despite the dog and boat change their positions, or $x_{CM} = constant$

$$\frac{m_1 x_{1i} + m_2 x_{2i}}{m_1 + m_2} = \frac{m_1 x_{1f} + m_2 x_{2f}}{m_1 + m_2}$$

Using notations from the diagram

$$m_1(d+L) + m_2\left(d + \frac{L}{2}\right) = m_1(d+x) + m_2\left(d + x + \frac{L}{2}\right)$$

4. Solving the last equation for x

$$x = \frac{m_1}{m_1 + m_2} L$$

Note that the answer does not depend on our choice of the origin. It was easier to set the origin at the initial position of the left side of the boat and adding the distance d later.

So, the dog's position relative to the shore is

$$d + x = d + \frac{m_1}{m_1 + m_2} L$$

5. Calculations

Note that $d = 6.1\ m - 2.4\ m = 3.7\ m$

$$d + x = 3.7\ m + \frac{4.5\ kg}{4.5\ kg + 18\ kg} 2.4\ m = 4.18\ m$$

6. Looking back.

Units are correct. The distance is not too short or too large.

For infinitely heavy boat $m_2 \gg m_1$ we get $x \approx 0$ and $d + x = 3.7\ m$ that corresponds to common sense.

Example 8-8

The cat Tom, of mass 7.0 kg, and the mouse Jerry (see the cartoon "Tom and Jerry") are in a 1.0 kg canoe. When the canoe is at rest in the placid water, they exchange seats, which are 2.0 m apart and symmetrically located with respect to the canoe's center. The canoe moves 1.7 m relative to the shore during the exchange. What is Jerry's mass?

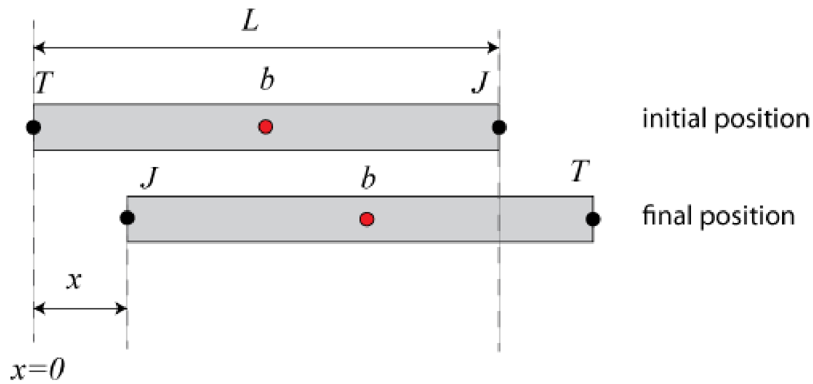
SOLUTION

1. Physics – system of "particles" (Tom, Jerry, the boat)

2. The basic equation

$$Ma_{CM} = \vec{F}^{ext}$$

3. Since there are no external forces, initially the system was at rest, and we have one dimensional case, then



$$x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i = constant$$

or despite all the changes the center of mass stays at the same position.

$$\frac{m_T x_{Ti} + m_J x_{Ji} + m_b x_{bi}}{m_T + m_J + m_b} = \frac{m_T x_{Tf} + m_J x_{Jf} + m_b x_{bf}}{m_T + m_J + m_b}$$

Using notations from the diagram, and assuming that the boat has moved to the right

$$m_T 0 + m_J L + m_b \frac{L}{2} = m_T(L + x) + m_J x + m_b\left(\frac{L}{2} + x\right)$$

4. Solving the last equation for m_J

$$m_J(L - x) = m_T(L + x) + m_b x$$

$$m_J = \frac{m_T(L + x) + m_b x}{L - x}$$

5. Calculations

$$m_J = \frac{7\,kg(2\,m + 1.7\,m) + 1\,kg * 1.7\,m}{2\,m - 1.7\,m} = 92\,kg$$

6. Looking back.

Units are correct but the answer does not have sense! We expect Jerry to be less than Tom. It means that our assumption that the boat has moved to the right was not correct. If Tom was heavier than Jerry, then the boat had to move to the left. Thus we should rather have negative x

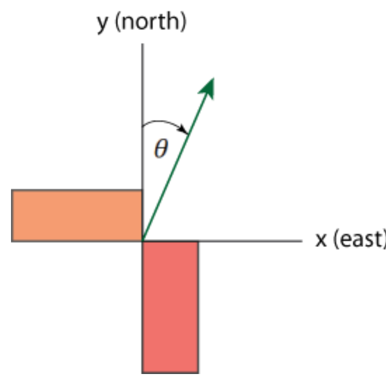
$$m_J = \frac{m_T(L-x) + m_b(-x)}{L+x}$$

$$m_J = \frac{7\,kg(2\,m - 1.7\,m) - 1\,kg * 1.7\,m}{2\,m + 1.7\,m} = 0.108\,kg$$

Now it looks reasonable.

<u>Example 8-9</u>

At an intersection, a car of mass 1500 kg going east collides with a pickup truck with mass 1800 kg is travelling north and ran a red light. The two vehicles stick together as a result of the collision and the wreckage slides 16.0 meters in a straight line 24^0 east of north. The coefficient of kinetic friction for the tires on the road is 0.90.



Calculate the speed of each vehicle before the collision.

SOLUTION:

1. Physics – completely inelastic collision in two dimensions+ dissipation of kinetic energy through friction (conservation of energy).

2. The basic equations

Completely inelastic collision (in vector form)

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

Conservation of energy (all kinetic energy after the collision goes into termal/friction)

$$\frac{1}{2}(m_1 + m_2)v_f^2 = \mu_k(m_1 + m_2)gd$$

3. Using components for velocities (where 1 is the car, and 2 is the truck)

$$x - component \quad m_1 v_{1i} + 0 = (m_1 + m_2)v_f \sin\theta$$
$$y - component \quad 0 + m_2 v_{2i} = (m_1 + m_2)v_f \cos\theta$$

4. From conservation of energy we can easily find the final speed of the wreckage

$$v_f = \sqrt{2\mu_k g d}$$

Using it in the momentum equation

$$v_{1i} = \frac{(m_1 + m_2)}{m_1} v_f \sin\theta = \frac{(m_1 + m_2)}{m_1}\left(\sqrt{2\mu_k g d}\right)\sin\theta$$
$$v_{2i} = \frac{(m_1 + m_2)}{m_2} v_f \cos\theta = \frac{(m_1 + m_2)}{m_2}\left(\sqrt{2\mu_k g d}\right)\cos\theta$$

5. Calculations

$$v_{1i} = 15\ m/s \qquad or\ \ 34\,mph$$
$$v_{2i} = 28\ m/s \qquad or\ \ 63\,mph$$

6. Looking back.

Units are correct. Both speeds are possible (not too fast or too slow) for cars.

*add example on rocket motion (with gravity)

9 Rotation in two dimensions

9.1 Rotational motion

In the previous chapters, we have been studying the mechanics of points, or small particles whose internal structure does not concern us. For the next two chapters we shall study the application of Newton's laws to more complicated things. When the world becomes more complicated, it also becomes more interesting, and we shall find that the phenomena associated with the mechanics of a more complex object than just a point are really quite striking. Of course these phenomena involve nothing but combinations of Newton's laws, but it is sometimes hard to believe that only $\vec{F} = m\vec{a}$ is at work.

The more complicated objects we deal with can be of several kinds: water flowing, galaxies whirling, and so on. The simplest "complicated" object to analyze, at the start, is what we call a rigid body, a solid object that is turning as it moves about. However, even such a simple object may have a most complex motion, and we shall therefore first consider the simplest aspects of such motion, in which an extended body rotates about a fixed axis. A given point on such a body then moves in a plane perpendicular to this axis. Such rotation of a body about a fixed axis is called plane rotation or rotation in two dimensions.

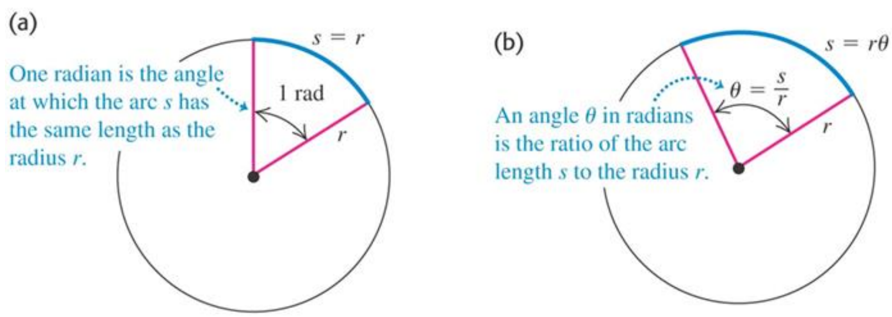
Of course an ordinary object does not simply rotate, it wobbles, shakes, and bends, so to simplify matters we shall discuss the motion of a nonexistent ideal object which we call a rigid body. This means object in which the forces between the atoms are so strong, that the little forces that are needed to move it do not bend it. Its shape stays essentially the same as it moves about. If we wish to study the motion of such a body and agree to ignore the motion of its center of mass, there is only one thing left for it to do, and that is to turn.

We have to describe that. How? Suppose there is some line in the body which stays put (perhaps it includes the center of mass and perhaps not), and the body is rotating about this particular line as an axis. How do we define the rotation? That is easy enough, for if we mark a point somewhere on the object, anywhere except on the axis, we can always tell exactly where the object is, if we only know where this point has gone to. The only thing needed to describe the position of that point is an angle. So rotation consists of a study of the variations of the angle with time.

9.2 Rotational variables

In order to study rotation, we observe the angle through which a body has turned. Of course, we are not referring to any particular angle inside the object itself; it is not that we draw some angle on the object. We are talking about the angular change of the position of the whole thing, from one time to another.

In describing rotational motion, the most natural way to measure the angle θ is not in degrees or revolutions but in *radians*.



Then

$$\theta = \frac{s}{r} \quad or \quad s = \theta r$$

Because the circumference of a circle of radius r is $2\pi r$, there are 2π radians in a complete circle

$$1 \text{ rev} = 360^0 = \frac{2\pi r}{r} = 2\pi \text{ rad}$$

and thus

$$1 \text{ rad} = \frac{360^0}{2\pi} = \frac{180^0}{\pi} \approx 57.3^0 \qquad 1 \text{ rad} = \frac{1}{2\pi} \approx 0.159 \text{ rev}$$

We do not reset θ to zero for each complete rotation of the reference line about the rotational axis. If the reference line completes two revolutions from the zero angular position, then the angular position θ of the line is $\theta = 4\pi$ rad.

First, let us study the kinematics of rotations. The angle will change with time, and just as we talked about position and velocity in one dimension, we may talk about angular position and angular velocity in plane rotation. In fact, there is a very interesting relationship between rotation in two dimensions and one-dimensional displacement, in which almost every quantity has its analog. First, we have the angle θ which defines how far the body has gone around; this replaces the distance x, which defines how far it has gone along. In the same manner, we have a velocity of turning, $\omega = d\theta/dt$, which tells us how much the angle changes in a second, just as $v = ds/dt$ describes how fast a thing moves, or how far it moves in a second. If the angle is measured in radians, then the angular velocity ω will be so and so many radians per second. The greater the angular velocity, the faster the object is turning, the faster the angle changes. We can go on: we can differentiate the angular velocity with respect to time, and we can call $\alpha = d\omega/dt = d^2\theta/dt^2$ the angular acceleration. That would be the analog of the ordinary acceleration.

9.2.1 Angular displacement

If a body rotates about the rotational axis, changing the angular position of the reference line from θ_1 to θ_2, then the body undergoes an *angular displacement* $\Delta\theta$ given by

$$\Delta\theta = \theta_2 - \theta_1 \tag{9.1}$$

The definition of angular displacement holds not only for the rigid body as a whole but also for every particle within that body because all the particles are locked together.

If a body is in translational motion along an x axis, its displacement Δx is either positive or negative, depending on whether the body is moving in the positive or negative direction of the axis. Similarly, the angular displacement $\Delta\theta$ of a rotating body is either positive or negative, according to the following rule:

An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.

9.2.2 Angular velocity

Suppose that our rotating body is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2. We define the *average angular velocity* of the body in the time interval Δt from t_1 to t_2 to be

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{9.2}$$

in which $\Delta\theta$ is the angular displacement that occurs during Δt (ω is the Greek letter omega).

The *(instantaneous) angular velocity* ω, with which we shall be most concerned, is the limit of the above ratio as Δt approaches zero. Thus,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \tag{9.3}$$

If we know $\theta(t)$, we can find the angular velocity ω by differentiation.

Equations for ω_{avg} and ω hold not only for the rotating rigid body as a whole but also *for every particle of that rigid body* because the particles are all locked together.

The unit of angular velocity is commonly the radian per second (rad/s) or the revolution per second (rev/s). Another measure of angular velocity is used in engineering, namely rpm, meaning number of revolutions per minute.

Two useful conversions are

$$1\ rev/s = 2\pi\ rad/s \quad and \quad 1\ rev/min = 1\ rpm = \frac{2\pi}{60} rad/s.$$

If a particle moves in translation along an x axis, its linear velocity v is positive or negative, depending on whether the particle is moving in the positive or negative direction of the axis. Similarly, the angular velocity ω of a rotating rigid body is either positive or negative, depending on whether the body is rotating counterclockwise (positive) or clockwise (negative).

The magnitude of an angular velocity is called the angular speed, which is also represented with ω.

9.2.3 Angular acceleration

If the angular velocity of a rotating body is not constant, then the body has an angular acceleration. Let ω_2 and ω_1 be its angular velocities at times t_2 and t_1, respectively. The *average angular acceleration* of the rotating body in the interval from t_1 to t_2 is defined as

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}, \tag{9.4}$$

in which $\Delta\omega$ is the change in the angular velocity that occurs during the time interval Δt. The *(instantaneous) angular acceleration α*, with which we shall be most concerned, is the limit of this quantity as Δt approaches zero. Then,

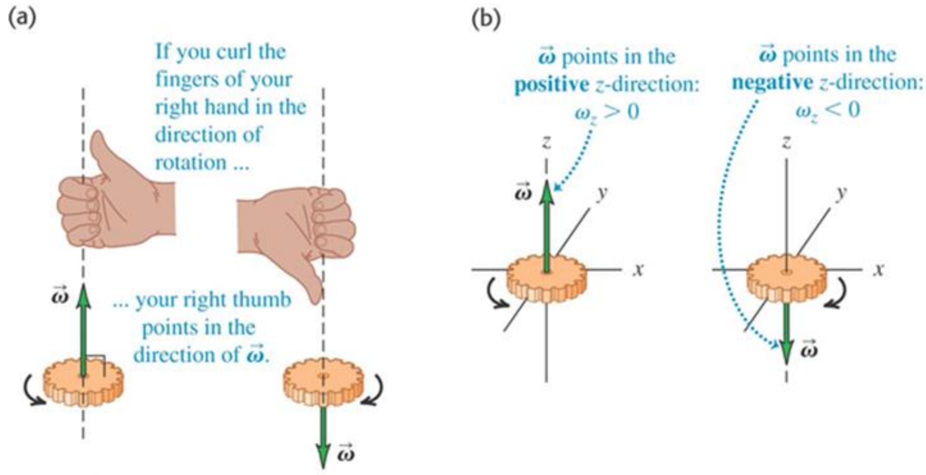
$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}. \tag{9.5}$$

Equations above hold not only for the rotating rigid body as a whole but also every particle of that body. The unit of angular acceleration is commonly the radian per second-squared (rad/s^2) or the revolution per second-squared (rev/s^2).

9.2.4 Angular velocity and angular acceleration as vectors

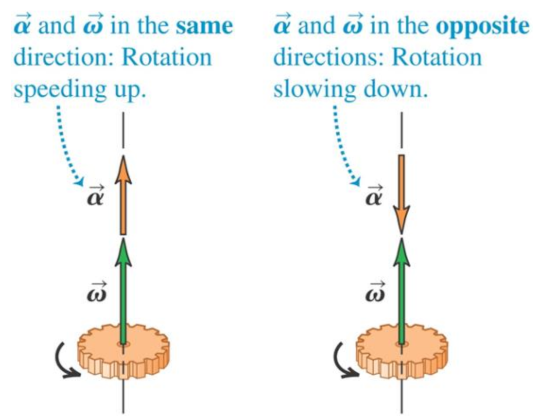
We can describe the position, velocity, and acceleration of a single particle by means of vectors. If a particle is confined to a straight line, however, we do not really need vector notation. Such a particle has only two directions available to it, and we can indicate these directions with plus and minus signs.

In the same way, a rigid body rotating about a fixed axis can rotate only clockwise or counterclockwise as seen along the axis, and again we can select between the two directions by means of plus and minus signs. However, in general the direction of angular velocity $\vec{\omega}$ (as a vector) is defined using a *right-hand rule* as figure shows



It is not easy to get used to representing angular quantities as vectors. We instinctively expect that something should be moving along the direction of a vector. That is not the case here. Instead, something (the rigid body) is rotating around the direction of the vector. In the world of pure rotation, a vector defines an axis of rotation, not a direction in which something moves. Furthermore, it obeys all the rules for vector manipulation discussed before.

Just as we did for angular velocity, it's useful to define an angular acceleration vector $\vec{\alpha}$



Now for the caution: Angular displacements (unless they are very small) cannot be treated as vectors. Why not? We can certainly give them both magnitude and direction as we did for the angular velocity vector. However, to be represented as a vector, a quantity must also obey the rules of vector addition, one of which says that if you add two vectors, the order in which you add them does not matter. Angular displacements fail this test.

9.3 Rotation with constant angular acceleration

In pure translation, motion with a constant linear acceleration (for example. that of a falling body) is an important special case. In pure rotation, the case of constant angular acceleration is also important, and a parallel set of equations hold for this case also. We shall not derive them here, but simply write them from the corresponding linear equations, substituting equivalent angular quantities for the linear ones.

Comparison of Linear and Angular Motion with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a = \text{constant}$	$\alpha = \text{constant}$
$v = v_0 + a_x t$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2} a t^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$
$v^2 = v_0{}^2 + 2a (x - x_0)$	$\omega^2 = \omega_0{}^2 + 2\alpha (\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v + v_0) t$	$\theta - \theta_0 = \frac{1}{2}(\omega + \omega_0) t$

9.4 Relating the linear and angular variables

In Section 3-5, we discussed uniform circular motion, in which a particle travels at constant linear speed v along a circle and around an axis of rotation. When a rigid body, such as a merry-go-round, rotates around an axis, each particle in the body moves in its own circle around that axis. Since the body is rigid, all the particles make one revolution in the same amount of time; that is, they all have the same angular speed ω. However, the farther a particle is from the axis, the greater the circumference of its circle is, and so the faster its linear speed v must be.

We often need to relate the linear variables s, v, and a for a particular point in a rotating body to the angular variables θ, ω, and α for that body. The two sets of variables are related by r, the perpendicular distance of the point from the rotation axis. This perpendicular distance is the distance between the point and the rotation axis, measured along a perpendicular to the axis. It is also the radius r of the circle traveled by the point around the axis of rotation.

9.4.1 The position

If a reference line on a rigid body rotates through an angle θ, a point within the body at a position r from the rotation axis moves a distance s along a circular arc, where s is given by

$$s = \theta r \tag{9.6}$$

This is the first of our linear-angular relations.

Caution: The angle θ here must be measured in radians because equation above is itself the definition of angular measure in radians.

9.4.2 The speed

Differentiating equation above with respect to time – with r held constant-leads to

$$\frac{ds}{dt} = \frac{d\theta}{dt} r$$

However, ds/dt is the linear speed (the magnitude of the linear velocity) of the point in question, and $d\theta/dt$ is the angular speed ω of the rotating body. So

$$v = \omega r \tag{9.7}$$

Caution: The angular speed ω must be expressed in radian measure.

This equation tells us that since all points within the rigid body have the same angular speed ω, points with greater radius r have greater linear speed v. (Remember that the linear velocity is always tangent to the circular path of the point in question). If the angular speed ω of the rigid body is constant, the linear speed v of any point within it is also constant. Thus, each point within the body undergoes uniform circular motion.

The period of revolution T for the motion of each point and for the rigid body itself is given by

$$T = \frac{2\pi r}{v}.$$

This equation tells us that the time for one revolution is the distance $2\pi r$ traveled in one revolution divided by the speed at which that distance is traveled. Substituting for v from $v = \omega r$ and canceling r, we find also that

$$T = \frac{2\pi}{\omega}$$

This equivalent equation says that the time for one revolution is the angular distance 2π rad traveled in one revolution divided by the angular speed (or rate) at which that angle is traveled.

9.4.3 The acceleration

Differentiating $v = \omega r$ with respect to time (again with r held constant) leads to

$$\frac{dv}{dt} = \frac{d\omega}{dt} r$$

Here we run up against a complication. Here dv/dt represents only the part of the linear acceleration that is responsible for changes in the magnitude v of the linear velocity $\vec{v}$. Like $\vec{v}$, that part of the linear acceleration is tangent to the path of the point in question. We call it the tangential component at of the linear acceleration of the point, and we write

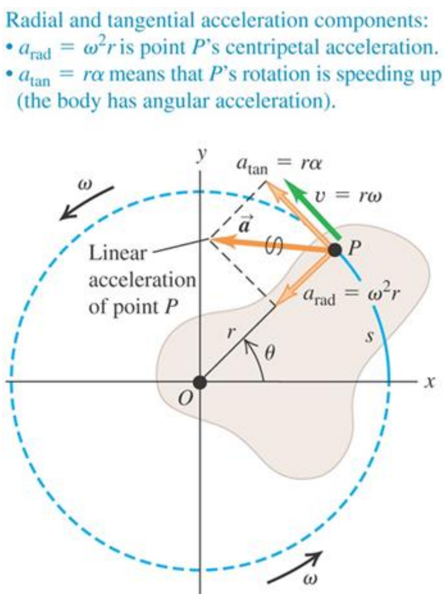
$$a_t = \alpha r \tag{9.8}$$

where $\alpha = d\omega/dt$. Caution: The angular acceleration α must be expressed in radian measure.

However, a particle (or point) moving in a circular path has a radial component of linear acceleration, $a_r = v^2/r$ (directed radially inward), that is responsible for changes in the direction of the linear velocity $\vec{v}$. By substituting for $v = \omega r$, we can write this component as

$$a_r = \frac{v^2}{r} = \omega^2 r. \tag{9.9}$$

Thus, the linear acceleration of a point on a rotating rigid body has, in general, two components.



The radially inward component a_r (given by $a_r = \omega^2 r$) is present whenever the angular velocity of the body is not zero. The tangential component a_t (given by $a_t = \alpha r$) is present whenever the angular acceleration is not zero.

The total linear acceleration vector of the point is

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

($\vec{a}_t$ describes the change in how fast the point is moving, and $\vec{a}_r$ represents the change in its direction of travel.) Because $\vec{a}$ is a vector having a radial and a tangential component, the magnitude of a for the point P on the rotating rigid object is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{\alpha^2 r^2 + \omega^4 r^2} = r\sqrt{\alpha^2 + \omega^4}$$

9.5 Kinetic energy of rotation

A rotating rigid body certainly has kinetic energy due to that rotation. How can we express that energy? We cannot apply the familiar formulae $K = (1/2)mv^2$ to the body as a whole because that would only give us the kinetic of the body's center of mass. (This is zero for pure rotation)

Instead, we shall treat a rotating rigid body as a collection of particles with different speeds. We can then add up the kinetic energies of all the particles to find the kinetic energy of the rotation.

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \cdots = \sum \frac{1}{2} m_i v_i^2$$

in which m_i is the mass of the i-th particle and v_i is its speed. The sum is taken over all the particles in the body. The problem with this equation is that v_i is not the same for all particles. We solve this problem by substituting $v_i = \omega r_i$

$$K = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

in which ω is the same for all particles of the rigid body.

The quantity in parentheses on the right side of equation above tells us how the mass of the rotating body is distributed about its axis of rotation. We call that quantity the *rotational inertia* (or *moment of inertia*) I of the body with respect to the axis of rotation. It is a constant for a particular rigid body and a particular rotation axis. (That axis must always be specified if the value of I is to be meaningful.) We may now write

$$I = \sum m_i r_i^2 \tag{9.10}$$

and then

$$K = \frac{1}{2} I\omega^2. \tag{9.11}$$

Because we have used the relation $v_i = \omega r_i$ in deriving, ω must be expressed in radian measure. The SI unit for I is the kilogram-square meter (kg·m²).

Equation above gives the kinetic energy of a rigid body in pure rotation. It is the angular equivalent of the formula $K = (1/2)Mv_{cm}^2$ which gives the kinetic energy of a rigid body in pure translation. In both formulas there is a factor of 1/2. Where mass M appears in one equation, I (which involves hoth mass and its distribution) appears in the other. Finally, each equation contains as a factor the square of a speed- translational or rotational as appropriate. The kinetic energies of translation and of rotation are not different kinds of energy. They are both kinetic energy expressed in ways that are appropriate to the motion at hand.

It is important that you recognize the analogy between kinetic energy associated with linear motion $(1/2)mv^2$ and rotational kinetic energy $(1/2)I\omega^2$. The quantities I and ω in rotational motion are analogous to m and v in linear motion, respectively. (In fact, I takes the place of m every time we compare a linear-motion equation with its rotational counterpart.) The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion. Note, however, that mass is an intrinsic property of an object, whereas I depends on the physical arrangement of that mass.

9.6 Calculating the rotational inertia

If a rigid body consists of a few particles we can calculate its rotational inertia about a given rotation axis with $I = \sum m_i r_i^2$. If a rigid body consists of a great many adjacent particles (it is continuous, like a Frisbee), using the sum is not practical. Thus, instead, we replace the sum with an integral and define the rotational inertia of the body as

$$I = \int r^2 dm \tag{9.12}$$

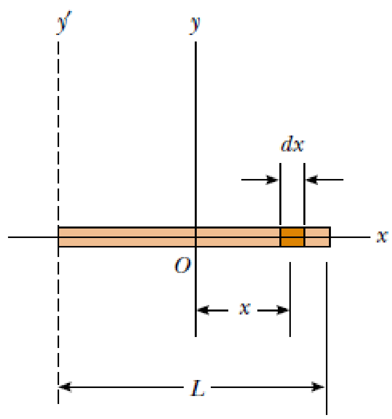
where dm represents the mass of any infinitesimal particle of the body and r is the perpendicular distance of this particle from the axis of rotation. The integral is taken over the whole body. This is easily done only for bodies of simple geometric shape. Since we can write $dm = \rho dV$ where ρ is the density, and dV is the elementary volume

$$I = \int r^2 \rho dV = \int \rho r^2 dV = \int \rho r^2 dxdydz.$$

For a body with a uniform density $\rho = \rho_0$ we have

$$I = \rho_0 \int r^2 dxdydz \tag{9.13}$$

These equation can be easily modified for linear objects ($dm = \lambda dl$) and flat objects ($dm = \sigma dS$).



Lets' calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod (the y axis) and passing through its center (the center of mass).

$$dm = \lambda dx = \frac{M}{L} dx$$

$$I_y = \int r^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \frac{M}{L} dx = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$I_y = \frac{1}{12} M L^2$$

Since students (in university physics I) have rather limited or no experience with multiple (two and three- dimensional integration) we will use results of such integration for most common bodies (see Figure 25).

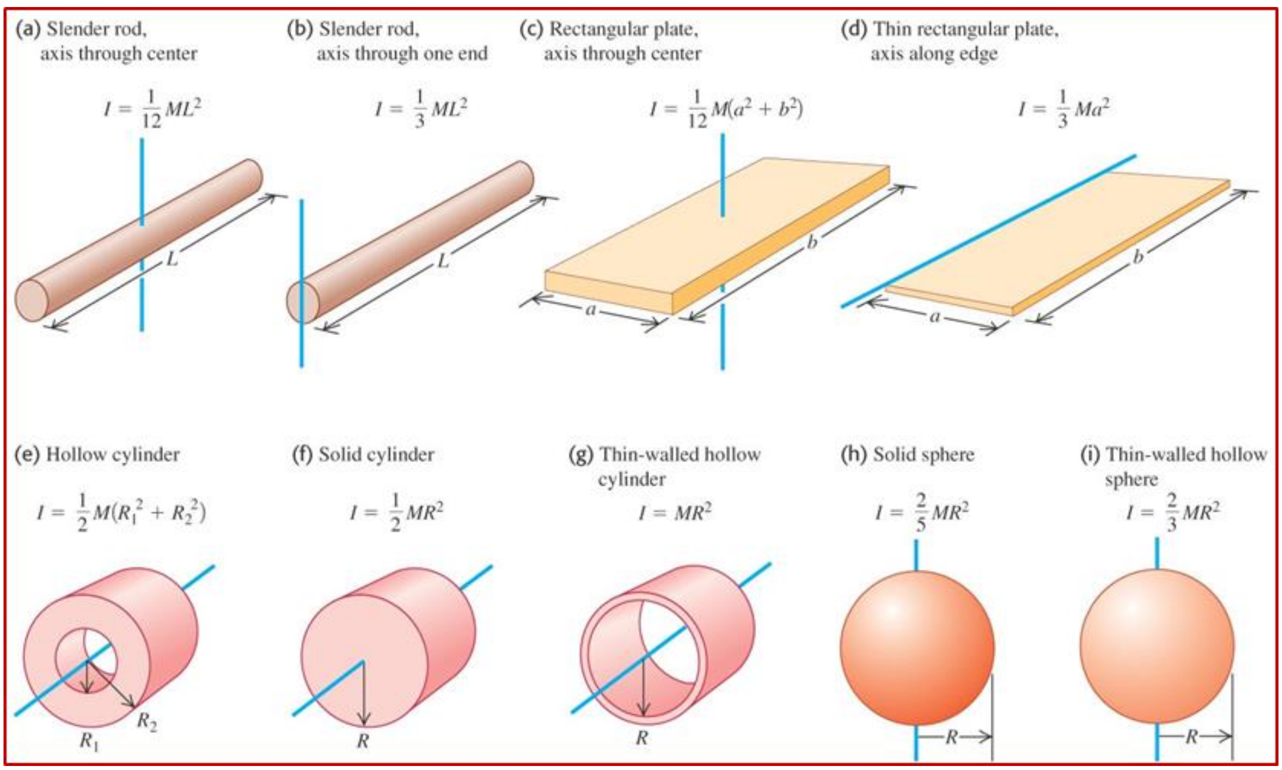


Figure 25 Moment of inertia of most common bodies

Suppose we want to find the rotational inertia I of a body of mass M about a given axis. In principle, we can always find I with the integration. However, there is a shortcut if we happen to already know the rotational inertia I_{cm} of the body about a parallel axis that extends through the body's center of mass. Let h be the perpendicular distance between the given axis and the axis through the center of mass (remember these two axes must be parallel). Then the rotational inertia I about the given axis is

$$I = I_{cm} + mh^2$$

This equation is known as the *parallel-axis theorem*.

9.7 Potential energy of a rigid body

Gravitational potential energy of a rigid body can be written as an infinite sum over all mass elements m_i as

$$U = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + \cdots = (m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots) g$$

However, having the definition for the center of mass (in y −direction)

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

together with definition for the total mass $M = m_1 + m_2 + m_3 + \cdots,$ one has

$$m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots = M y_{cm}$$

Combining this with the previous equation for U, we get

$$U = Mgy_{cm} \tag{9.14}$$

where y_{cm} is the vertical coordinate of the center of mass.

9.8 Examples

Example 9-1

The angular velocity of helicopter blades (a rotor) changes from 320 rev/min to 225 rev/min in 5.0 s as the rotor slows down to rest.

a) What time interval is required for the blades to come to rest from their initial angular velocity of 320 rev/min?
b) How many revolutions do the blades make in coming to rest from the initial 320 rev/min?

SOLUTION:

1. Physics – rotation with constant angular acceleration

2. The basic equations

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega = \omega_0 + \alpha t$$

3. Let's note that the angular acceleration for the problem can be found from the given information (change for angular velocity from ω_1 to ω_2 in Δt time interval), namely

$$\alpha = \frac{\omega_2 - \omega_1}{\Delta t}$$

The first unknown (the time to stop) can be found from the second equation of motion ($\omega = \omega_0 + \alpha t$) using the condition $\omega_f = 0$, and the second question can be easily answered using the first equation of motion with constant angular acceleration

4. Let t_f be the time required to stop. Then

$$t_f = \frac{\omega_f - \omega_0}{\alpha} = -\frac{\omega_0}{\alpha}$$

Then the total angular displacement during this time is

$$\theta = \theta_0 + \omega_0 t_f + \frac{1}{2}\alpha t_f^2$$

5. Calculations

We better work with SI units. Then

$$320\frac{rev}{min} = 320\frac{rev}{min}\left(\frac{2\pi\ rad}{rev}\right)\left(\frac{1\ min}{60\ s}\right) = 33.5\ \frac{rad}{s},$$

$$225\frac{rev}{min} = 225\frac{rev}{min}\left(\frac{2\pi\ rad}{rev}\right)\left(\frac{1\ min}{60\ s}\right) = 23.6\ \frac{rad}{s}$$

$$\alpha = \frac{(23.6 - 33.5)\ rad/s}{5\ s} = -1.98\ rad/s^2$$

$$t_f = -\frac{33.5\ rad/s}{-1.98\ rad/s^2} = 16.8\,s$$

$$\theta = 0 + 33.5\frac{rad}{s}\cdot 16.8\,s + \frac{1}{2}(-1.98\ rad/s^2)\cdot(16.8\,s)^2 = 282\,rad \quad or \quad \frac{282\,rad}{2\pi} = 45\,rev.$$

6. Looking back.

Units are correct. Both time and number of revolutions seems reasonable.

<u>Example 9-2</u>

What is the linear speed of a point (in mph)

a) on Earth's equator?
b) on northern pole?
c) at latitude of Norfolk? (coordinates of Norfolk are about 36°N and 76°W)

SOLUTION:

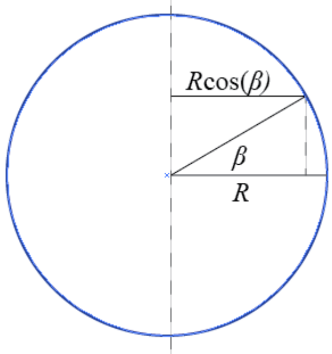
SOLUTION 1 (the easiest)

By definition speed = distance/time. Let's consider circular motion with radius R and period (time) T. Then distance = circumference = $2\pi R$. And speed on Earth's equator ($R = R_E$)

$$v = \frac{2\pi R}{T} = \frac{2\cdot 3.1415\cdot 6.37\cdot 10^6 m}{24\ h} = 1.667\cdot 10^6\frac{m}{h} = 1.667\cdot 10^6\frac{m}{h}\left(\frac{1\ mile}{1609\ m}\right) = 1036\,mph$$

Since rotational radius depend on the latitude as $R = R_E\cos\theta$ we can easily calculate speeds for Norfolk VA and the Northern pole.

SOLUTION 2 (connecting angular and linear variables)



1. Physics – connecting rotational and translational variables.

2. The basic equations

$$v = \omega R$$

3. We know that the one revolution $\theta = 2\pi$ of the planet takes 24 hours. Then the angular speed

$$\omega = \frac{2\pi}{T}$$

and the linear speed

$$v = \omega R = \frac{2\pi}{T}R$$

4. The radius of revolution depends on latitude as $R = R_E\cos\beta$ (e.g. angle $\beta = 36^0$ for Norfolk). Thus

$$v = \frac{2\pi R_E}{T}\cos\beta$$

5. Calculations for angles 0^0 (Equator), 36^0 (Norfolk, VA), and 90^0 (Northern Pole)

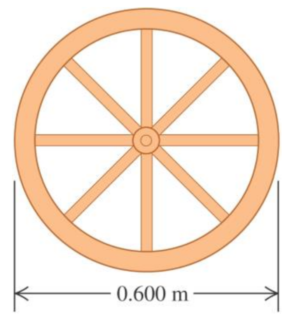
$$v_{equator} = 1036\, mph, \qquad v_{Nrfolk} = 839\, mph, \qquad v_{Pole} = 0\, mph$$

6. Looking back.

Units are correct.

Example 9-3

A wagon wheel is constructed as shown. The radius of the wheel is 0.300 m, and the rim has mass 1.40 kg. Each of the eight spokes that lie along the diameter are 0.300 m long has mass 0.280 kg.



What is the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel?

SOLUTION:

1. Physics – calculating rotational inertia for a complex object.

2. The basic equations

$$I = \sum m_i r_i^2$$

$$I = \rho \int r^2 dxdydz$$

3. The problem looks quite complicated because we cannot apply the first equation for the solid body, and certainly we do not want to integrate over the complex shape. However, any rotational inertia for an object that has several parts can be written as

$$I_{total} = I_1 + I_2 + \cdots I_n$$

4. The wheel can be viewed as the rim + eight identical spokes, then

$$I_{total} = I_{rim} + 8 I_{spoke}$$

From Figure 25 we have $I_{rim} = M_{rim} R_{rim}^2$ and $I_{spoke} = (1/3) m_{spoke} R_{spoke}^2$

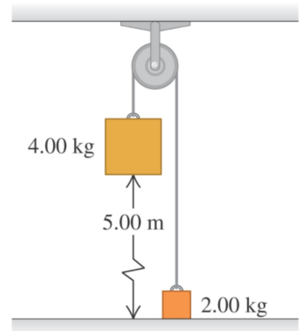
5. Calculations

$$I_{total} = 1.4\, kg\ \cdot (0.30\, m)^2 + 8 \cdot (1/3) \cdot 0.28\ \mathrm{kg}\ \cdot (0.30\, m)^2 = 0.193\ \mathrm{kg \cdot m^2}$$

6. Looking back.

Units are correct.

Example 9-4



The pulley (a solid cylinder) has radius 0.160 m and mass of 1.0 kg. The rope does not slip on the pulley rim. Use energy methods to calculate the speed of the 4.0-kg block just before it strikes the floor.

SOLUTION:

1. Physics – conservation of energy for translational and rotational motion in presence of gravitational force.

2. The basic equations

$$K_i + U_i = K_f + U_f$$

$$K_t = \frac{1}{2}mv^2, \quad K_r = \frac{1}{2}I\omega^2, \quad U_g = mgh$$

3. There is a system with three connected parts, so we have to include every part into equation for conservation of energy.

Let's use notations m for 2,0 kg block, M for 4.0 kg block, and I for the pulley. Then

$$K_{mi} + U_{gmi} + K_{Mi} + U_{gMi} + K_{Ii} + U_{gIi} = K_{mf} + U_{gmf} + K_{Mf} + U_{gMf} + K_{If} + U_{gIf}$$

4. Initially the system is at rest, so all initial kinetic energies are zero. We are going to count the gravitational potential energy from the ground. Let's also note that gravitational potential energy of the pulley does not change. With this in mind we have

$$U_{gMi} = K_{mf} + U_{gmf} + K_{Mf} + K_{If}$$

or being more specific

$$Mgh = \frac{1}{2}mv^2 + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + mgh$$

where h=5.0 m.

Since the rope does not slip, then $v = \omega R$. For a solid cylinder $I = 0.5M_cR^2$. Then we can rewrite the last equation as

$$(M - m)gh = \frac{1}{2}mv^2 + \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}M_cR^2\right)\frac{v^2}{R^2} = \frac{1}{2}v^2\left(m + M + \frac{1}{2}M_c\right)$$

Solving for v gives

$$v = \sqrt{\frac{2(M - m)gh}{m + M + \frac{1}{2}M_c}}$$

5. Calculations

$$v = \sqrt{\frac{2(4.0\ kg - 2.0\ kg)9.8\ \ m/s^2 \cdot 5\ m}{(2.0 + 4.0 + 0.5 \cdot 1.0)\ kg}} = 5.5\ m/s$$

6. The units are correct. For $M_c = 0$ we get the answer for example 7-7.

10 Dynamics of rotational motion

10.1 Torque

Let us now move on to consider the dynamics of rotation. Here a new concept, force, must be introduced. Let us inquire whether we can invent something which we shall call the torque (Latin *torquere*, to twist) which bears the same relationship to rotation as force does to linear movement. A force is the thing that is needed to make linear motion, and the thing that makes something rotate is a "rotary force" or a "twisting force," i.e., a torque. Qualitatively, a torque is a "twist". What is a torque quantitatively?

Here is some *intuitive* approach to define torque. A doorknob is located as far as possible from the door's hinge line for a good reason. If you want to open a heavy door, you must certainly apply a *force*: that alone, however, is not enough. *Where you apply that force* and in what *direction* you push are also important.

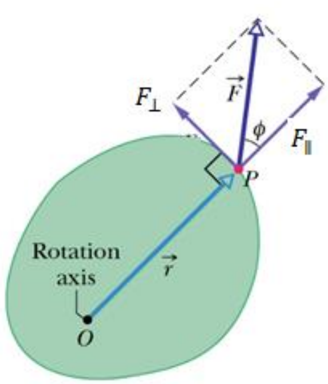


Figure shows a cross section of a body that is free to rotate about an axis passing through O and perpendicular to the cross section. A force $\vec{F}$ is applied at point P, whose position relative to O is defined by a position vector $\vec{r}$. The directions of vectors $\vec{F}$ and $\vec{r}$ make an angle ϕ with each other. (For simplicity, we consider only forces that have no component parallel to the rotation axis: thus $\vec{F}$ is in the plane of the page.).

To determine how $\vec{F}$ results in a rotation of the body around the rotation axis, we resolve $\vec{F}$ into two components. One component, called the parallel (radial) component $F_{\|}$ points along $\vec{r}$. This component does not cause rotation because it acts along a line that extends through O. (If you pull on a door parallel to the plane of the door, you do not rotate the door.) The other component of $\vec{F}$, called the perpendicular (tangential) component $F_{\perp}$, is perpendicular to $\vec{r}$ and has magnitude $F_{\perp} = F \sin\phi$. This component does cause rotation. (If you pull on a door perpendicular to its plane, you can rotate the door.)

The ability of $\vec{F}$ to rotate the body depends not only on the magnitude of its tangential component $F_{\perp}$, but also on just how far from O the force is applied. To include both these factors, we define a quantity called torque τ as the product of the two factors and write it as

$$\tau = rF\sin\phi = rF_{\perp}$$

Torque, which comes from the Latin word meaning "to twist," may be loosely identified as the turning or twisting action of the force $\vec{F}$. When you apply a force to an object such as a screwdriver or torque wrench with the purpose of turning that object, you are applying a torque.

The SI unit of torque is the newton-meter (N·m). *Caution:* The newton-meter is also the unit of work. Torque and work however, are quite different quantities and must not be confused. Work is often expressed in joules (1 J = 1 N·m), but torque never is.

Now we can write a general definition for torque based on a vector product of two vectors, namely

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.1}$$

This definition has both magnitude and direction because torque is a vector.

At this point we need to talk more about vector product of two vectors

10.2 Vector Product

The vector product of two vectors $\vec{A}$ and $\vec{B}$, also called a cross product, is denoted by $\vec{A} \times \vec{B}$. The vector product is itself a vector.

$$\vec{C} = \vec{A} \times \vec{B} \tag{10.2}$$

The *magnitude* of the vector (cross) product is defined as

$$C = AB\sin\phi \tag{10.3}$$

We measure the angle ϕ from $\vec{A}$ toward $\vec{B}$, and take it to be the smallest of two possible angles, so ϕ ranges from 0^0 to 180^0 (the magnitude of C is always positive)

Some properties of the cross product

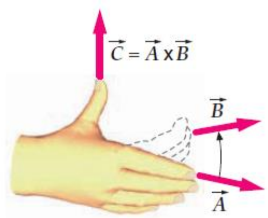
$$\vec{A} \times \vec{A} = 0$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$\frac{d}{dt}\left(\vec{A}\times\vec{B}\right)=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}.$$

There are always two directions perpendicular to a given plane, one on each side of a plane. The *direction* of the vector product follows the right-hand rule.



Imagine rotating vector $\vec{A}$ about the perpendicular line until it is align with $\vec{B}$, choosing the smaller of the two possible angles between $\vec{A}$ and $\vec{B}$. Curl the fingers of your right hand around the perpendicular line so that the fingertips point in the direction of rotation; your thumb will then point in the direction of $\vec{A}\times\vec{B}$. Figure shows this right-hand rule

If we know the components of $\vec{A}$ and $\vec{B}$, we can calculate the components of the vector product $\vec{C}=\vec{A}\times\vec{B}$ using properties of unit vectors

$$\hat{\imath}\times\hat{\imath}=\hat{\jmath}\times\hat{\jmath}=\hat{k}\times\hat{k}=0$$

$$\begin{gathered}\hat{\imath}\times\hat{\jmath}=-\hat{\jmath}\times\hat{\imath}=\hat{k}\\ \hat{\jmath}\times\hat{k}=-\hat{k}\times\hat{\jmath}=\hat{\imath}\\ \hat{k}\times\hat{\imath}=-\hat{\imath}\times\hat{k}=\hat{\jmath}\end{gathered}$$

we get

$$\vec{C}=\vec{A}\times\vec{B}=\left(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}\right)\times\left(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}\right)$$

$$=\left(A_yB_z-A_zB_y\right)\hat{\imath}+\left(A_zB_x-A_xB_z\right)\hat{\jmath}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

Thus, the components of the resulting vector are given by

$$\begin{aligned}C_x&=A_yB_z-A_zB_y\\ C_y&=A_zB_x-A_xB_z\\ C_z&=A_xB_y-A_yB_x\end{aligned}\tag{10.4}$$

10.3 Torque as a vector product

The torque $\vec{\tau}$ acting on acting on the particle relative to the fixed point 0 is a vector quantity

$$\vec{\tau}=\vec{r}\times\vec{F}$$

We can evaluate the vector (or cross) product in this definition of by using

$$\tau=rF\sin\phi=rF_{\perp}\tag{10.5}$$

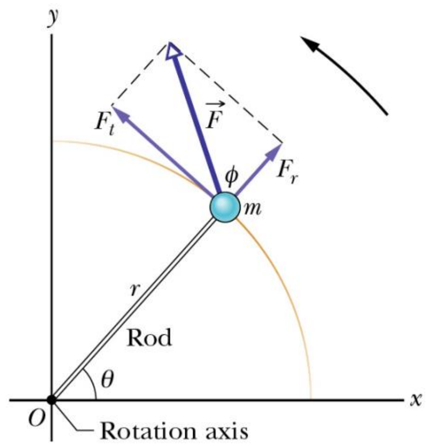
or in the component form (keeping in mind that normally we place $\vec{r}$ and $\vec{F}$ in the *xy*-plane),

$$\tau_z=r_xF_y-r_yF_x\tag{10.6}$$

10.4 Newton’s Second Law for rotation

A torque can cause rotation of a rigid body, as when you use a torque to rotate a door.

Let's consider the simple situation shown in figure to the left.



The rigid body there consists of a particle of mass *m* on one end of a massless rod of length *r*. The rod can move only by rotating about its other end, around a rotation axis (an axle) that is perpendicular to the plane of the page. Thus, the particle can move only in a circular path that has the rotation axis at its center.

A force $\vec{F}$ acts on the particle. However, because the particle can move only along the circular path, only the tangential component F_t of the force (the component that is tangent to the circular path) can accelerate the particle along the path. We can relate F_t to the particle's tangential acceleration at along the path with Newton's second law, writing

$$F_t = ma_t$$

The torque acting on the particle is

$$\tau = F_t r = ma_t r$$

From $a_t = \alpha r$ we can write this as

$$\tau = m(\alpha r)r = (mr^2)\alpha$$

The quantity in parentheses on the right side of equation above is the rotational inertia of the particle ($I = mr^2$) about the rotation axis. Thus,

$$\tau = I\alpha$$

We can easily extend this equation to any rigid body rotating about a fixed axis, because any such body can always be analyzed as an assembly of single particles.

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = (I_1 + I_2 + \cdots)\alpha_z = I\alpha_z$$

Thus we can write rotational analog of Newton's second law for a *rigid* body (We assume that the angular acceleration α is the same for all particles in the body). For rotations in *xy*-plane

$$\sum \tau_z = \tau_{net,z} = I\alpha_z \tag{10.7}$$

Generally in vector form for rotation in three dimensional space

$$\vec{\tau} = I\vec{\alpha} \tag{10.8}$$

Just as Newton's second law says that the net force on a particle equals the particle's mass times its acceleration, equation above says that the net torque on a rigid body equals the body's moment of inertia about the rotation axis times its angular acceleration. Thus, this equation is the rotational analog of Newton's second law for linear motion $\vec{F} = m\vec{a}$.

This equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces. According to Newton's third law, the internal forces that any pair of particles in the rigid body exerts on each other are equal and opposite. If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite and add to zero. Hence all the internal torques add to zero, so the sum in $\tau_{net,z}$ includes only the torques of the external forces.

10.5 Rolling

We can extend our analysis of the rotational motion to some cases in which the axis of rotation moves. When that happens, the motion of the body is combined translation and rotation. The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of translational motion of the center of mass and rotation about an axis through the center of mass. This is true even when the center of mass accelerates, so that it is not at rest in any inertial frame.

Consider a system with two particles (extension to n particles is evident). The kinetic energy K of a system of particles is the sum of the kinetic energies of the individual particles:

$$K = K_1 + K_2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

The position of a particle can be written as the sum of two vectors, the position of the center of mass and the position of the particle relative to the center of mass

$$\vec{r} = \vec{r}_{cm} + \vec{s}$$

Then differentiating this vector we get the velocity of a particle as

$$\vec{v} = \vec{v}_{cm} + \vec{u}$$

where $\vec{v}_{cm}$ is the velocity of the center of mass and $\vec{u}$ is the velocity of the particle relative to the center of mass. Substituting it into equation for the kinetic energy, we obtain

$$\begin{aligned}
K &= \frac{1}{2}m_1(\vec{v}_{cm} + \vec{u}_1)^2 + \frac{1}{2}m_2(\vec{v}_{cm} + \vec{u}_2)^2 \\
&= \frac{1}{2}m_1 v_{cm}^2 + \frac{1}{2}m_2 v_{cm}^2 + \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 + m_1\vec{v}_{cm}\vec{u}_1 + m_2\vec{v}_{cm}\vec{u}_2 \\
&= \frac{1}{2}(m_1 + m_2)v_{cm}^2 + \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 + \vec{v}_{cm}(m_1\vec{u}_1 + m_2\vec{u}_2)
\end{aligned}$$

The quantity $m_1\vec{u}_1 + m_2\vec{u}_2$ is equal to

$$m_1\vec{u}_1 + m_2\vec{u}_2 = \frac{d}{dt}(m_1\vec{s}_1 + m_2\vec{s}_2) = \frac{d}{dt}\big((m_1 + m_2)\vec{s}_{cm}\big)$$

where $\vec{s}_{cm}$ is the position of the center of mass relative to the center of mass, and clearly $\vec{s}_{cm} = 0$. Then $m_1\vec{u}_1 + m_2\vec{u}_2 = (m_1 + m_2)\vec{u}_{cm} = 0$ (the velocity of anything relative to itself is always equal to zero.) It follows that

$$K = \frac{1}{2}(m_1 + m_2)v_{cm}^2 + \frac{1}{2}(m_1u_1^2 + m_2u_2^2)$$

The last term is the sum of the kinetic energies calculated by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation with the same angular speed $\omega = u/r$, then

$$\frac{1}{2}(m_1u_1^2 + m_2u_2^2) = \frac{1}{2}(m_1\omega^2r_1^2 + m_2\omega^2r_2^2) = \frac{1}{2}(m_1r_1^2 + m_2r_2^2)\omega^2 = \frac{1}{2}(I_1 + I_2)\omega^2$$

Thus, the kinetic energy is the sum of a part associated with motion of the center of mass and a part associated with rotation about an axis through the center of mass:

$$K = \frac{1}{2}(m_1 + m_2)v_{cm}^2 + \frac{1}{2}(I_1 + I_2)\omega^2$$

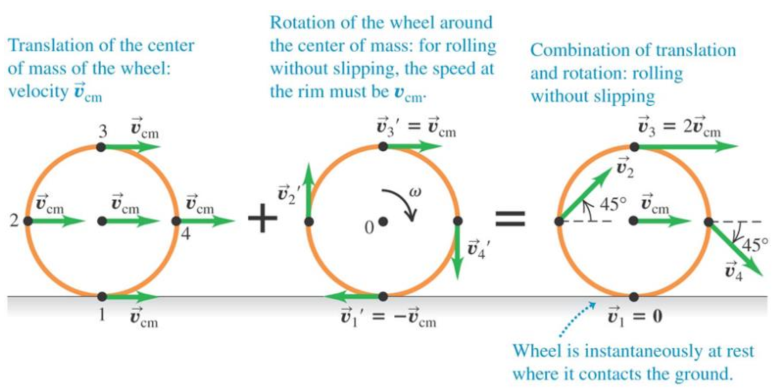
For rigid bodies the rotational inertia does not change during motion. Thus, for any system of particles or a rigid body we can write

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \tag{10.9}$$

where M is the total mass of the system, and I_{cm} is the rotational inertia associated with rotation about an axis through the center of mass.

10.5.1 Smooth rolling

An important case of combined translation and rotation is *rolling without slipping*, such as the motion of a wheel. The wheel is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously at rest so that it does not slip. Hence the velocity $\vec{v}_1'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{cm}'$. If the radius of the wheel is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v}_{cm}'$ is $R\omega$; hence we must have $v_{cm} = R\omega$ for smooth (without slipping) rolling motion.



As the figure above shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus, while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward twice as fast as the center of mass, and points 2 and 4 at the sides have velocities at 45^0 to the horizontal.

Now, in the kinetic energy of rolling

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

we have

$$v_{cm} = \omega R \qquad (10.10)$$

or the linear and angular velocities are not independent.

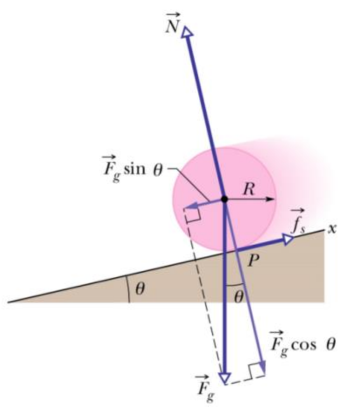
10.5.2 The forces of rolling

If a wheel rolls at constant speed it has no tendency to slide at the point of contact P, and thus no frictional force acts there. However, if a net force acts on the rolling wheel to speed it up or to slow it down, then that net force causes acceleration a_{cm} of the center of mass along the direction of travel. It also causes the wheel to rotate faster or slower, which means it causes an angular acceleration α about the center of mass. These accelerations tend to make the wheel slide at P. Thus, a frictional force must act on the wheel at P to oppose that tendency.

If the wheel *does not slide*, the force is a static frictional force $\vec{f_s}$ and the motion is smooth rolling. We can then relate the magnitudes of the linear acceleration and the angular acceleration by differentiating $v_{cm} = \omega R$ with respect to time (with R held constant). So, for smooth rolling we have

$$a_{cm} = \alpha R$$

Let's study rolling down a ramp.



This figure shows a round uniform body of mass M and radius R rolling smoothly down a ramp at angle θ, along an x axis. We want to find expressions for the body's acceleration a_{cm} down the ramp. We do this by using Newton's second law in both its linear version ($F_{net} = Ma$) and its angular version ($\tau_{net} = I\alpha$).

We start by drawing the forces on the body

1. The gravitational force $\vec{F_g}$ on the body is directed downward. The tail of the vector is placed at the center of mass of the body. The component along the ramp is $f_g \sin\theta$, which is equal to $Mg \sin\theta$.

2. A normal force $\vec{N}$ is perpendicular to the ramp. It acts at the point of contact P, but the vector has been shifted along its direction until its tail is at the body's center of mass.

3. A static frictional force $\vec{f_s}$ acts at the point of contact P and is directed up the ramp. (Do you see why? If the body were to slide at P, it would slide down the ramp. Thus, the frictional force opposing the sliding must be up the ramp.) We can write Newton's second law for components along the x axis in

$$f_s - Mg\sin\theta = Ma_{cm}.$$

This equation contains two unknowns, f_s and a_{cm}. We should not assume that f_s is at its maximum value $f_{s,max}$. All we know is that the value of f_s, is just right for the body to roll smoothly down the ramp, without sliding.

We now wish to apply Newton's second law in angular form to the body's rotation about its center of mass $\tau_{net} = I\alpha$. The frictional force produces a torque Rf_s, which is positive because it tends to rotate the body counterclockwise in Figure. Forces$\vec{F_g}$ and $\vec{N}$ produce zero torques (do you see why?). So we can write the angular form of Newton's second law about an axis through the body's center of mass as

$$Rf_s = I_{cm}\alpha.$$

This equation contains two unknowns, f_s and α.

Because the body is rolling smoothly, we can use $a_{cm} = \alpha R$ to relate the unknown accelerations. But we must be cautious because here a_{cm} is negative (in the negative direction of the x axis) and α is positive (counterclockwise). Thus we substitute $-a_{cm}/R$ for α, and

$$f_s = -I_{cm}\frac{a_{cm}}{R^2}$$

Substituting it to the first equation $f_s - Mg\sin\theta = Ma_{cm}$ we find

$$a_{cm} = -\frac{g\sin\theta}{1 + I_{cm}/MR^2}$$

We can use this equation to find the linear acceleration of any body rolling smoothly (without slipping) along an incline.

For most rolling objects the rotational inertia I can be written as $I = \beta MR^2$ where β is a coefficient specific for specific shape (for example $\beta = 2/5$ for a solid sphere). Then

$$a_{cm} = -\frac{g\sin\theta}{1+\beta}$$

and

$$f_s = \frac{\beta}{1+\beta}Mg\sin\theta$$

Let’s find the static frictional force needed to prevent slipping

$$\mu_s Mg\cos\theta \geq \frac{\beta}{1+\beta}Mg\sin\theta$$

or

$$\mu_s \geq \frac{\beta}{1+\beta}\tan\theta.$$

10.6 Translation and rotation dynamics

Rolling is a nice example of combined translation and rotation dynamics. Earlier we demonstrated (section 8.3) that for a body with total mass M, the acceleration of the center of mass is the same as that of a point mass M acted on by all the external forces on the actual body:

$$\sum \vec{F}^{ext} = M\vec{a}_{CM}$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7)

$$\sum \vec{\tau}_z = I_{cm}\vec{\alpha}_z$$

provided the following two conditions are met for combined translation and rotation motion

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation. We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, if the rotation axis satisfies the two conditions just mentioned

10.7 Work and Power in Rotational Motion

In chapter 6 we related change in kinetic energy of a moving particle with force acting on that particle along some path

$$\Delta K = K_f - K_i = \int_i^f \vec{F}\cdot d\vec{r} = W(i \to f)$$

Then the power was defined as

$$P = \frac{dW}{dt} = \vec{F}\cdot\vec{v}$$

We can easily derive similar equations for a rotational situation. Here are just resulting equations

$$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \int_i^f \tau_z d\theta = W(i \to f)$$

When τ_z is constant

$$\Delta K = \tau_z(\theta_f - \theta_i) = W(i \to f)$$

The rotational equivalent of power for rotational motion is

$$P = \frac{dW}{dt} = \tau_z \omega_z$$

10.8 Angular momentum

Although we have only considered the special case of a rigid body thus far, the properties of torques and their mathematical relationships are interesting also even when an object is not rigid. In fact, we can prove a very remarkable theorem: just as external force is the rate of change of a quantity $\vec{p}$, which we call the total momentum of a collection of particles, the external torque is the rate of change of a quantity $\vec{L}$ which we call the angular momentum of the group of particles.

To prove this, we shall suppose that there is a system of particles on which there are some forces acting and find out what happens to the system as a result of the torques due to these forces. First, of course, we should consider just one particle of mass m moving around an axis O; the particle is not necessarily rotating in a circle about O, it may be moving in an ellipse, like a planet going around the sun, or in some other curve. The torque is

$$\tau_z = xF_y - yF_x$$

and using Newton's second law we can write it as

$$\tau_z = xm\frac{d^2y}{dt^2} - ym\frac{d^2x}{dt^2}$$

Now, although this does not appear to be the derivative of any simple quantity, it is in fact the derivative of the quantity

$$xm\left(\frac{dy}{dt}\right) - ym\left(\frac{dx}{dt}\right)$$

$$\frac{d}{dt}\left[xm\left(\frac{dy}{dt}\right) - ym\left(\frac{dx}{dt}\right)\right] = xm\left(\frac{d^2y}{dt^2}\right) + \left(\frac{dx}{dt}\right)m\left(\frac{dy}{dt}\right) - ym\left(\frac{d^2x}{dt^2}\right) - \left(\frac{dy}{dt}\right)m\left(\frac{dx}{dt}\right)$$
$$= xm\frac{d^2y}{dt^2} - ym\frac{d^2x}{dt^2}$$

So it is true that the torque is the rate of change of something with time! So we pay attention to the "something," we give it a name: we call it L, the angular momentum, where for z −component

$$L_z = xm\left(\frac{dy}{dt}\right) - ym\left(\frac{dx}{dt}\right) = xp_y - yp_x$$

So we have found that there is also a rotational analog for the momentum, and that this analog, the angular momentum, is given by an expression in terms of the components of linear momentum that is just like the formula for torque in terms of the force components!

Generally, in vector notation the angular momentum for a particle with constant mass m, velocity $\vec{v}$ momentum $\vec{p}$ and position vector $\vec{r}$ relative to the origin O of an inertial frame is defined as

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{10.11}$$

Like torque, angular momentum depends upon the position of the axis about which it is to be calculated.

10.8.1 Conservation of angular momentum

Now we shall go on to consider what happens when there is a large number of particles, when an object is made of many pieces with many forces acting between them and on them from the outside. Now suppose we add the torques $\vec{\tau}_i$ for all the particles and call it the total torque $\vec{\tau}$. Now it might seem that the total torque is a complicated thing. There are all those internal forces and all the outside forces to be considered. But, if we take Newton's law of action and reaction to say, not simply that the action and reaction are equal, but also that they are directed exactly oppositely along the same line, then the two torques on the reacting objects, due to their mutual interaction, will be equal and opposite because the lever arms for any axis are equal.

Let's consider the total angular momentum

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

Differentiating with respect to t

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}_1 \times \vec{p}_1) + \frac{d}{dt}(\vec{r}_2 \times \vec{p}_2)$$

Because

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}$$

then

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}_1 \times \vec{p}_1) + \frac{d}{dt}(\vec{r}_2 \times \vec{p}_2) = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

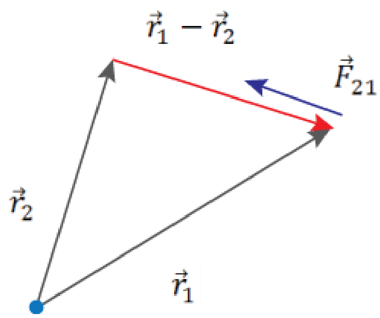
Now let's separate the effect of the internal and external forces.

$$\vec{F}_1 = \vec{F}_{net,1} = \vec{F}_{21} + \vec{F}_1^{ext}, \ \ \vec{F}_2 = \vec{F}_{net,2} = \vec{F}_{12} + \vec{F}_2^{ext}$$

Substituting it into equation before, we find that

$$\frac{d}{dt}\vec{L} = \vec{r}_1 \times \vec{F}_{21} + \vec{r}_1 \times \vec{F}_1^{ext} + \vec{r}_2 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_2^{ext}$$

Let's concentrate on the internal forces where $\vec{F}_{21} = -\vec{F}_{12}$ (action-reaction).



$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = \vec{r}_1 \times \vec{F}_{21} - \vec{r}_2 \times \vec{F}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21}$$

According to the vector diagram, vectors $\vec{r}_1 - \vec{r}_2$ and $\vec{F}_{21}$ are collinear and their cross product is zero.

Therefore, the internal torques balance out pair by pair, and we have the remarkable theorem that the rate of change of the total angular momentum about any axis is equal to the external torque about that axis!

$$\vec{\tau} = \sum \vec{\tau}_i = \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \tag{10.12}$$

Thus we have a very powerful theorem concerning the motion of large collections of particles, which permits us to study the over-all motion without having to look at the detailed machinery inside. This theorem is true for any collection of objects, whether they form a rigid body or not.

One extremely important case of the above theorem is the law of conservation of angular momentum: if no external torques acts upon a system of particles, the angular momentum remains constant.

A special case of great importance is that of a rigid body, that is, an object of a definite shape that is just turning around. Consider an object that is fixed in its geometrical dimensions, and which is rotating about a fixed axis. For an object going around in a circle, the angular momentum is

$$L_i = m_i v_i r_i = m_i r_i^2 \omega$$

or, summing over all the particles i, we get

$$\vec{L} = I\vec{\omega} \tag{10.13}$$

where i is the moment of inertia. This is the analog of the law that the momentum is mass times velocity. There is one important difference between mass and moment of inertia which is very dramatic. The mass of an object never changes, but its moment of inertia can be changed. If we stand on a frictionless, rotatable stand with our arms outstretched, and hold some weights in our hands as we rotate slowly, we may change our moment of inertia by drawing our arms in, but our mass does not change. If the external torque is zero, then the angular momentum remains constant.

$$I_1\vec{\omega}_1 = I_2\vec{\omega}_2 \tag{10.14}$$

Quite often in this course we consider rotational motion on xy plane, and then we can write

$$I_1\omega_{1z} = I_2\omega_{2z} \tag{10.15}$$

That is, if we reduce the moment of inertia, we have to increase the angular velocity.

10.9 Examples

Example 10-1

While exploring the castle, Exena the Exterminator is spotted by a dragon that chases her down a hallway. Exena runs into a safe room and attempts to swing a heavy door shut before the dragon gets her. The door is initially perpendicular to the wall, so it must be turned through 90^0 to be closed. The door is 3.00 m tall and 1.25 m wide, and its mass is 200.0 kg. If Exena applies a force of 220 N at the edge of the door and perpendicular to it, how much time does it take her to shut the door? Ignore the friction in the hinges.

SOLUTION:

1. Physics – Newton's second law for rotational + motion with constant angular acceleration (because the force is constant)

2. The basic equations

$$\tau = I\alpha$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

3. For the door $I = (1/3)Mw^2$, where w is the width of the door. Then

$$F_E w = \frac{1}{3}Mw^2\alpha$$

$$\theta = \frac{\pi}{2} = \frac{1}{2}\alpha t^2$$

4. From the first equation

$$\alpha = \frac{3F_E}{Mw}, \quad t = \sqrt{\frac{2\theta}{\alpha}}$$

5. Calculations

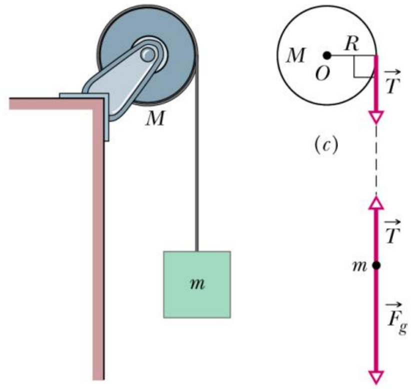
$$\alpha = \frac{3 \cdot 220\ N}{200.0\ kg \cdot 1.25\ m} = 2.64\ \frac{N}{kg \cdot m} = 2.64\frac{kg \cdot m \cdot s^{-2}}{kg \cdot m} = 2.64\frac{rad}{s^2}$$

$$t = \sqrt{\frac{2 \cdot \pi/2}{2.64\ rad/s^2}} = 1.09\ s$$

6. Units are correct both for the angular acceleration and time. It is difficult to apply common sense to fairy tales, but with the force of 220 N (or about 50 lb) applied to the massive door (about 440 lb) the time seems realistic.

<u>Example 10-2</u>

Figure shows a uniform disk, with mass M and radius R, mounted on a fixed horizontal axel. A block with mass m hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling disk, and the tension in the cord. The cord does not slip, and there is no friction in the axel.



SOLUTION:

1. Physics – Newton's second law for translational (the block) and rotational (the disk) motions.

2. The basic equations

$$F = ma$$

$$\tau = I\alpha$$

3. As always we must have a free-body diagram for every object in the picture. Besides we use for the disk $I = (1/2)MR^2$, and since the cord does not slip $a = \alpha R$.

4. For the falling block

$$T - mg = -ma$$

For the disk

$$-RT = -I\alpha = -\frac{1}{2}MR^2\frac{a}{R} = -\frac{1}{2}MRa$$

These two equations have two unknowns, namely T and a. From the second equation

$$T = \frac{1}{2}Ma$$

Using it in the first equation

$$\frac{1}{2}Ma - mg = -ma \qquad a\left(m + \frac{1}{2}M\right) = mg$$

$$a = \frac{2m}{2m + M}g$$

$$T = \frac{mM}{2m + M}g$$

5. Calculations

There were no data to calculate

6. Let's analyze the answer using our past knowledge.

It's clear that for massless disk the acceleration of the block must be . For $M = 0$ we get

$$a = \frac{2m}{2m + M} g = \frac{2m}{2m} g = g$$

For a very heavy disk $M \gg m$ we get $a \approx 0$ and $T \approx mg$ (also this is what we expected).

Example 10-3

A golf ball (mass of 45.93 g and diameter of 42.67 mm) starts from rest and rolls without slipping a distance of 12.0 m down a hill towards a pond. The hill is inclined at 5^0.

a) What fraction of the total kinetic energy of the golf ball is due to rotation as it gets to the pond?
b) What is the speed of the golf ball at the bottom of the hill?

SOLUTION:

1. Physics – rolling, conservation of energy

2. The basic equations

$$K_i + U_i = K_f + U_f$$

$$K_t = \frac{1}{2} m v^2, \quad K_r = \frac{1}{2} I \omega^2, \quad U_g = mgh$$

3. For the first question we need to find

$$\frac{K_r}{K_t + K_r}$$

and the second question can be solved using conservation of energy $U_i = K_t + K_r$

4. Additional information: a golf ball can be treated as a solid sphere, then $I = (2/5) m R^2$ and for rolling without slipping $v = \omega R$

$$K_r = \frac{1}{2}\left(\frac{2}{5} m R^2\right)\frac{v^2}{R^2} = \frac{1}{5} m v^2$$

$$\frac{K_r}{K_t + K_r} = \frac{\frac{1}{5} m v^2}{\frac{1}{2} m v^2 + \frac{1}{5} m v^2} = \frac{2}{7}$$

Note that the answer does not depend on the hill's incline or even shape and size.

Applying conservation of energy

$$mgh = \frac{1}{2} m v^2 + \frac{1}{5} m v^2 = \frac{7}{10} m v^2$$

where $h = d \sin\theta$ (d=12.0 m, and $\theta = 5^0$)

$$v = \sqrt{\frac{10}{7} gh} = \sqrt{\frac{10}{7} g\, d\, sin\, \theta}$$

5. Calculations

The ratio has already being calculated as 2/7.

$$v = \sqrt{\frac{10}{7} 9.8\ m/s^2 \cdot 12\, m \cdot \sin 5^0} = 3.8\, m/s$$

6. Units are correct. The speed seems reasonable. By the way, we did not need data for the golf ball, except the shape.

<u>Example 10-4</u>

One warm April day, Tom and Jerry went hiking. On a quiet road going straight down a hill they found a rusty and abandoned Jeep Wrangler. They decided to compete in rolling various objects (the fastest object wins). Jerry took a tire (you may consider a tire as a hoop, with a diameter of 0.80 m and mass of 5.0 kg). Tom had no other choice but a custom made wheel (a solid cylinder with a diameter of 0.60 m and mass of 20.0 kg). Both the tire and the wheel started from rest and rolled down half a mile without slipping. The incline of the road was 5.0^0.

Was Jerry good in physics?

Hint: you may answer this question if you can find what object was the first at the bottom of the hill, or what object had higher speed at the end (higher speed = less time).

SOLUTION:

1. Physics – rolling, conservation of energy

2. The basic equations

$$K_i + U_i = K_f + U_f$$

$$K_t = \frac{1}{2} mv^2, \quad K_r = \frac{1}{2} I\omega^2, \quad U_g = mgh$$

3. For a rolling object we have

$$0 + mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2.$$

Rotational inertia can be written as

$$I = \beta m R^2$$

where $\beta = 1$ for a hoop, and $\beta = 1/2$ for a cylinder. Since $\omega = v/R$, then

$$\frac{1}{2}I\omega^2 = \frac{1}{2}(\beta m R^2)\left(\frac{v^2}{R^2}\right) = \frac{1}{2}\beta m v^2$$

4. Now conservation of energy can be written as

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\beta m v^2 = \frac{1}{2}(1+\beta)mv^2.$$

$$v = \sqrt{\frac{2gh}{1+\beta}}$$

5. Calculations

Actually, we do not need to do any calculations. An object with larger β has less speed. $\beta_{hoop} > \beta_{cylinder}$. Therefore, Tom won the competition.

6. Looking back.

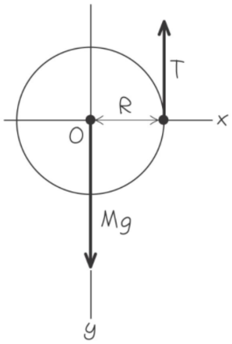
Units are correct. (This problem is very similar to example 10-3)

<u>Example 10-4</u>

While trying to escape, captain Jack Sparrow jumps off a bridge between two cliffs with a rope wrapped around his body (like a primitive yo-yo, see the picture). The bridge is 24.0 meters above the jungle and the cord is also 24.0 meters long.

Evaluate the speed of his center of mass v_{cm} before touching the ground. You may consider his body as a solid cylinder with $I = MR^2/2$, with radius 18.0 cm, and mass of 72 kg.

SOLUTION:



1. Physics – dynamics of rotational motion (since we have to deal with forces), one dimensional vertical motion with constant acceleration.

2. The basic equations for translational and rotational dynamics

$$F = ma$$

$$\tau = I\alpha$$

and for one dimensional (vertical) motion with constant acceleration

$$y = y_0 + v_0 t + \frac{at^2}{2}$$
$$v = v_0 + at$$

3. Our first goal is to find the acceleration for Jack using the free-body diagram

Translational motion: $T - Mg = -Ma$

Rotational motion: $TR = (1/2)MR^2\alpha$

4. Since $a = \alpha R$, then we use $\alpha = a/R$ in the rotational motion equation

$$TR = (1/2)MR^2\alpha = (1/2)MR^2\, a/R$$

or $T = (1/2)MR^2\alpha = (1/2)Ma$. Substituting this into the translational motion equation gives

$$\frac{1}{2}Ma - Mg = -Ma$$

$$a = \frac{2}{3}g, \qquad T = \frac{1}{3}Mg$$

Now, eliminating time from one dimension motion equations with constant acceleration gives $v^2 - v_0^2 = 2ay$ and for $v_0 = 0$ we get $v = \sqrt{2ay}$.

5. Calculations

$$v = \sqrt{2ay} = \sqrt{\frac{4}{3}gy} = \sqrt{1.333 \cdot 9.8\ m/s^2 \cdot 24\ m} = 17.7\ m/s$$

6. Units are correct. The speed is about 40 mph. That is a high speed to survive, but probably this is not a problem for Jack Sparrow.

SOLUTION 2:

Quite often the same problem can be solved by using rotational dynamics OR/AND conservation of energy. Let's solve the same problem but now using conservation of energy

1. Physics – conservation of energy with translational and rotational motion.

2. The basic equations for rotational dynamics

$$K_i + U_i = K_f + U_f$$

$$K_t = \frac{1}{2}mv^2, \quad K_r = \frac{1}{2}I\omega^2, \quad U_g = mgh$$

3. Conservation of energy with all terms

$$K_{ti} + K_{ri} + U_i = K_{tf} + K_{rf} + U_f$$

4. The body starts from rest (zero initial kinetic energy). We also can count the ground as zero potential energy level.

$$Mgy = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

The rotational inertia is $I = MR^2/2$. We also use that $\omega = v/R$ then

$$\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v^2}{R^2} = \frac{1}{4}Mv^2$$

Then conservation of energy can be written as

$$Mgy = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2$$

Solving for v gives

$$v = \sqrt{\frac{4}{3}gy}$$

Note that the same answer we got using rotational dynamics!

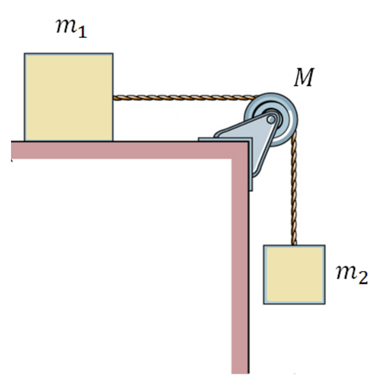
5. Calculations

$$v = \sqrt{\frac{4}{3}gy} = \sqrt{1.333 \cdot 9.8\ m/s^2 \cdot 24\ m} = 17.7\ m/s$$

6. The same result as before but using a different method.

Example 10-5

A block of mass m_1 and a block of mass m_2 are connected by a massless string over a pulley in the shape of a solid disk having radius R and mass M. The blocks are allowed to move as shown in figure. The coefficient of kinetic friction is μ_k. Determine



a) the acceleration of the two blocks

b) the tensions in the string on both sides of the pulley.

SOLUTION:

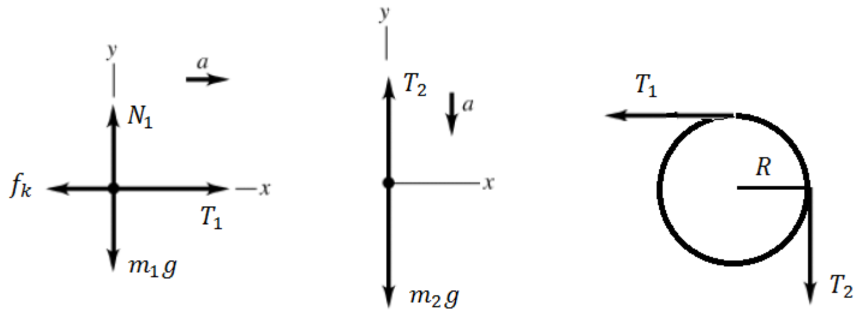
1. Physics – dynamics of translational and rotational motion (actually we can find acceleration using energy consideration but we must use rotational dynamics to find tensions).

2. Newton's second law for translational and rotational motion

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

3. We draw free body diagrams for EVERY object!



Using Newton's second law (attention – note proper positive or negative signs for forces and accelerations)

$$T_1 - f_k = m_1 a$$
$$N_1 - m_1 g = 0$$
$$T_2 - m_2 g = -m_2 a$$
$$RT_1 - RT_2 = -I\alpha$$

where

$I = \beta MR^2$ ($\beta = 1/2$ for a disk), $\alpha = a/R$ and $f_k = \mu_k N_1$. From the second equation $N_1 = m_1 g$.
4. The system can be written as

$$T_1 - \mu_k m_1 g = m_1 a$$
$$T_2 - m_2 g = -m_2 a$$
$$RT_1 - RT_2 = -\beta MR^2 a/R$$

The last equation can be simplified

$$T_1 - T_2 = -\beta Ma$$

Now we have a system of three equations with three unknowns. Using the tensions from the first two equations and substituting them the last equation gives

$$\mu_k m_1 g + m_1 a - m_2 g + m_2 a = -\beta Ma$$

or

$$m_1 a + m_2 a + \beta Ma = m_2 g - \mu_k m_1 g$$

and finally

$$a = \frac{m_2 - \mu_k m_1}{m_1 + m_2 + \beta M} g$$

Having the acceleration we can easily find the tensions T_1 and T_2

5. Calculations

The problem does not need calculations

6. We can analyze the problem for various "extreme" cases.

a) $\beta = 0$ and $\mu_k = 0$ then

$$a = \frac{m_2}{m_1 + m_2} g$$

Thus, we got the same answer as example 5-5 from chapter 5.

b) $\beta = 0$ (massless disk), then

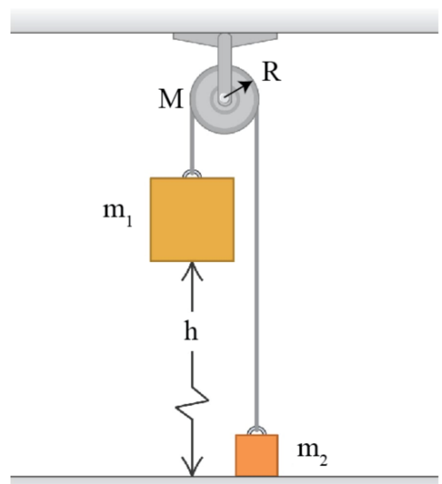
$$a = \frac{m_2 - \mu_k m_1}{m_1 + m_2} g$$

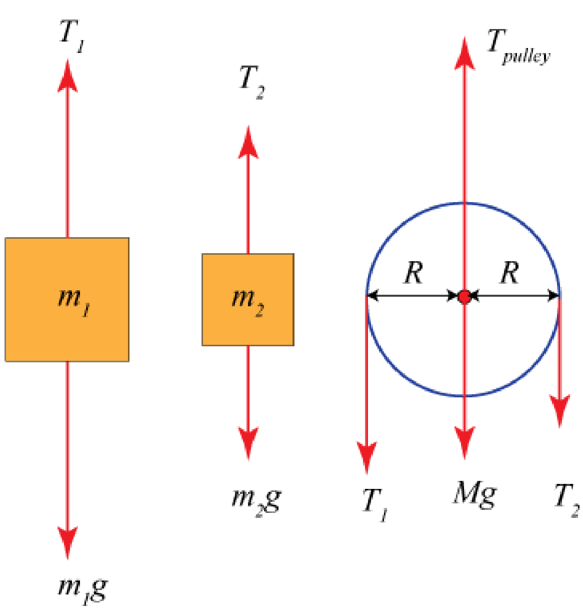
Now we have a solution for problem 6 from the first exam (spring 2013)

<u>Example 10-6</u>

Consider the system shown in Figure with m_1=20.0 kg, m_2=10.0 kg, R = 0.2 m, and the mass of the uniform pulley (cylinder) M = 5.0 kg. Object m_2 is resting on the floor, and object m_1 is h=4.0 meters above the floor when it is released from rest. The pulley axis is frictionless. The cord is light, does not stretch, and does not slip on the pulley.

a) What is the acceleration of the two blocks?
b) What are the tensions in the string on both sides of the pulley?
c) What the force does the ceiling exerts on the pulley?





SOLUTION:

1. Physics – dynamics of translational and rotational motion

(actually we can find acceleration using energy consideration but we must use rotational dynamics to find tensions).

2. Newton's second law for translational and rotational motion

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

3. We must have free body diagrams for EVERY object! Then we can write Newton's second law using components

$$F_{net,y1} = T_1 - m_1 g = -m_1 a \qquad \text{for mass 1}$$
$$F_{net,y2} = T_2 - m_2 g = m_2 a \qquad \text{for mass 2}$$
$$F_{net,yM} = T_{pulley} - Mg - T_1 - T_2 = 0 \quad \text{for pulley}$$
$$\tau_{net,z} = RT_1 - RT_2 = I\alpha \qquad \text{for pulley}$$

where $I = \beta MR^2$ ($\beta = 1/2$ for a disk), $\alpha = a/R$. Then the last equation can be written as

$$RT_1 - RT_2 = I\alpha = \frac{1}{2}MR^2\frac{a}{R} = \frac{1}{2}MRa, \qquad or \qquad T_1 - T_2 = \frac{1}{2}Ma$$

4. The system of equation for the problem

$$T_1 - m_1 g = -m_1 a$$
$$T_2 - m_2 g = m_2 a$$
$$T_{pulley} = Mg + T_1 + T_2$$
$$T_1 - T_2 = \frac{1}{2}Ma$$

Using the tensions from the first two equations in the torque equation gives

$$m_1 g - m_1 a - m_2 g - m_2 a = \frac{1}{2}Ma$$

or

$$m_1 g - m_2 g = m_1 a + m_2 a + \frac{1}{2}Ma = \left(m_1 + m_2 + \frac{1}{2}M\right)a$$

and finally solving for a

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M}g$$

Having the acceleration we can easily find the tensions T_1, T_2 and T_{pulley}

5. Calculations

$$a = \frac{(20-10)kg}{\left(20 + 10 + \frac{5}{2}\right)kg}9.8\ m/s^2 = 3.0\ m/s^2$$

$$T_1 = 136\ N, \qquad T_2 = 128\ N, \quad T_{pulley} = 313\ N$$

6. Looking back. Units are correct. Note that $T_{pulley} = 313\ N < m_1 g + m_2 g + Mg = 343\ N$

Example 10-7

Collapsing Spinning Star: The volume of a collapsing spinning star changes to 1/27 of its initial value without losing the mass.

a) What is the ratio of the new rotational inertia to the initial rotational inertia?
b) What is the ratio of the new rotational kinetic energy to the initial rotational kinetic energy?

Note: consider the star as a solid spherical object.

SOLUTION:

1. Physics – conservation of angular momentum, kinetic energy

2. The basic equations for rotational dynamics

$$I_i\omega_i = I_f\omega_f$$

$$K = \frac{1}{2}I\omega^2$$

3. We need to connect the volume of a sphere with its rotational inertia.

$$V = \frac{4}{3}\pi R^3$$

$$I = \frac{2}{5}MR^2$$

4. Using the given information

$$V_f = \frac{1}{27}V_i$$

$$\frac{4}{3}\pi R_f^3 = \frac{1}{27}\frac{4}{3}\pi R_i^3$$

then

$$R_f = \frac{1}{3}R_i$$

5. Since the rotational inertia is proportional to R^2 then

$$I_f = \frac{1}{9}I_i$$

from conservation of angular momentum follows

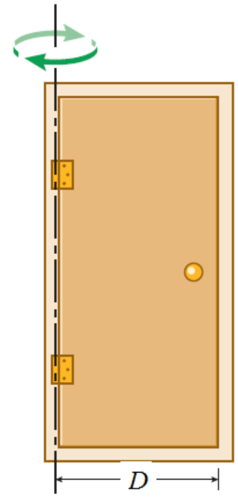
$$\omega_f = 9\omega_i$$

For their kinetic energies

$$K_f = \frac{1}{2}I_f\omega_f^2 = \frac{1}{2}\frac{1}{9}I_i(9\omega_i)^2 = 9K_i$$

6. It is clear that the final kinetic energy must be more than initial kinetic energy since some force pulled the matter closer to the center of rotation (normally it is gravity doing work).

<u>Example 10-8</u>



A door 1.0 m wide, of mass 25.0 kg, is hinged on one side so that it can rotate without friction. It is unlatched. A police office fires a bullet into the center of the door perpendicular to the center of the door (the mass and speed are 10 g and 400 m/s respectively).

a) Find the angular speed of the door just after the bullet embeds itself into the door.
b) Is kinetic energy conserved?
c) How much time will it take to have the door wide open (90 degrees)?

Note that the rotational inertia of a door is given by $\frac{1}{3}MD^2$

SOLUTION:

1. Physics – conservation of angular momentum, kinetic energy, motion with constant angular velocity.

2. The basic equations

$$\vec{L} = \vec{r} \times m\vec{v} \qquad \text{Angular momentum of a particle}$$
$$\vec{L} = I\vec{\omega} \qquad \text{Angular momentum of a rigid body}$$
$$K = \frac{1}{2}mv^2, \qquad K = \frac{1}{2}I\omega^2$$

3. Let's apply conservation of angular momentum

$$L_i = L_f$$

where

$$L_i = \frac{D}{2}mv_i \qquad \text{the bulet}$$
$$L_f = (I_{bullet} + I_{door})\omega_f \qquad \text{the bullet + door}$$
$$I_{bullet} = m\left(\frac{D}{2}\right)^2, \qquad I_{door} = \frac{1}{3}MD^2$$

4. Then all together

$$\frac{D}{2}mv_i = \left(\frac{1}{4}mD^2 + \frac{1}{3}MD^2\right)\omega_f$$
$$\omega_f = \frac{mv_i}{\left(\frac{1}{2}m + \frac{2}{3}M\right)D}$$

$$t = \frac{\Delta\theta}{\omega_f} = \frac{\pi/2}{\omega_f}$$

5. Calculations

$$\omega_f = 0.24\frac{rad}{s}, \quad t = 6.5\ s$$

$$K_i = 800\ J, \qquad K_f = 0.24\ J$$

6. Dimensions for all units are correct. The time seems accurate though it is much less than in Hollywood movies. The energy is not conserved, only a tiny fraction of energy goes into the rotation (that is consistent with results for the example 8-5)

11 Equilibrium

11.1 The conditions for equilibrium

So far we were dealing with motion. However, a state with no motion (equilibrium) has a great deal of interest as well. We are surrounded by objects that are not moving in any way – either in translation or in rotation – in the reference frame from which we observe them. Such objects are in *static equilibrium*.

The analysis of static equilibrium is particularly important in architecture and engineering. Such analysis is necessary to ensure that bridges do not collapse under there traffic and wind loads, building (even the tallest ones) are safe for work or living. However, design engineers are not the only ones who should do the equilibrium analysis.

The theory of equilibrium is based on Newton's laws of motion. A particle is in equilibrium, whenever the vector sum of all forces acting on it is zero

$$\vec{F}_{net} = 0.$$

For *extended bodies* this condition is not enough because they have a tendency to rotate. Therefore for extended objects the sum of torques about any point must be zero

$$\vec{\tau}_{net} = 0.$$

Therefore, the analysis of equilibrium is based on these two conditions. Despite the equations look awfully simple, applications to real problems can be challenging, and quite often require using computers for solving large system of equations.

Thus, the two requirements for a body to be in equilibrium are follows

1. The vector sum of all the external forces that act on the body, must be zero

$$\vec{F}_{net} = \sum \vec{F}_i = 0 \tag{11.1}$$

2. The vector sum of all the external torques that act on the body, measured about ANY possible point, must be zero

$$\vec{\tau}_{net} = \sum \vec{\tau}_i = 0 \tag{11.2}$$

These requirements obviously hold for static equilibrium. They also hold for the more general equilibrium in which total linear and angular momenta are constant but not zero. In the special case of *static equilibrium*, which is the main subject of this chapter, the object is at rest and so has no linear or angular speed (that is, $v_{cm} = 0,\ \omega = 0$).

The equilibrium conditions, as vector equations, can be written in the component form, one for each direction of the coordinate axis.

Balance of forces	Balance of torques
$F_{net.x} = 0$	$\tau_{net,x} = 0$
$F_{net,y} = 0$	$\tau_{net,y} = 0$
$F_{net,z} = 0$	$\tau_{net,z} = 0$

In introductory physics classes we normally consider situations in which the forces that act on the body are in *xy*-plane. That means that the only torques that act on the body can cause rotation around an axis parallel to the *z*-axis. Thus we eliminate one force equation, and two torques equations form the component equations, leaving

$$\begin{aligned} F_{net,x} &= 0 \qquad \text{(balance of forces along } x) \\ F_{net,y} &= 0 \qquad \text{(balance of forces along } y) \\ \tau_{net,z} &= 0 \qquad \text{(balance of torques around } z) \end{aligned} \tag{11.3}$$

Here $\tau_{net,z}$ is the net torque that the external forces produce either around z-axis or about any axis parallel to it.

If an object is in translational equilibrium and if the net torque is zero about one axis, then the net torque must also be zero about any other axis. The axis can be inside or outside the borders of the object. Consider an object being acted on by several forces

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = 0$$

Let $\vec{r}_1$ is the point of application of $\vec{F}_1$ relative to some point O, $\vec{r}_2$ is the point of application of $\vec{F}_2$ relative to the same point O and so on. Then, the net torque about an axis through O is

$$\vec{\tau}_{net} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots$$

Now consider another arbitrary point O' having a position vector $\vec{r}'$ relative to O. Then, the point of application of $\vec{F}_1$ relative to O' is identified by the vector $\vec{r}_1 - \vec{r}'$. Likewise we can write for other forces $\vec{F}_2$, $\vec{F}_3$ … Therefore, the net torque about an axis through O' is

$$\vec{\tau}_{net} = (\vec{r}_1 - r') \times \vec{F}_1 + (\vec{r}_2 - r') \times \vec{F}_2 + \cdots = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots - r'\left(\vec{F}_1 + \vec{F}_2 + \cdots\right)$$

Because the net force is assumed to be zero (given that the object is in translational equilibrium), the last term vanishes, and we see that the torque about O' is equal to the torque about O. Hence, if an object is in translational equilibrium and the net torque is zero about one point, then the net torque must be zero about any other point.

There are many interesting problems when an object is in rotational equilibrium but not in translational (an accelerating car, when the center of gravity is above the road). It is still possible to solve such problems but the conditions for equilibrium are different. In this case we deal with mechanics of non-inertial frames. This subject is commonly considered in courses of theoretical classical mechanics. Here we only show a way to approach such problems. For example, for translational motion with acceleration along x we have

$$\begin{aligned} F_{net,x} &= ma \qquad \text{(motion with acceleration along } x) \\ F_{net,y} &= 0 \qquad \text{(balance of forces along } y) \\ \tau_{net,z} &= 0 \qquad \text{(balance of torques around } z) \end{aligned}$$

Then we face a careful choice for a rotational axis because $\vec{F}_{net} \neq 0$. Normally such point is located at the center of mass (see example 11-9, 11-10).

11.2 The center of gravity

The gravitational force on an extended body is the vector sum of the gravitational forces acting on the individual elements (the atoms) of the body. Instead of considering all those individual elements, we can say: The gravitational force $\vec{F}_g$ on a body effectively acts at a single point, called the center of gravity of the body. Here the word "effectively" means that if the forces on the individual elements were somehow turned off and force $\vec{F}_g$ at the center of gravity were turned on, the net force and the net torque (about any point) acting on the body would not change.

First we review the definition for the center of mass (see section 8.3) for a group of particles in (x, y) plane with masses $m_1, m_2, \ldots$ and coordinates $(x_1, x_2, \ldots)$, $(y_1, y_2, \ldots)$ the center of mass is given

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

We wish to locate the center of gravity $(x_{cg},\ y_{cg})$, the point at which application of the single gravitational force $M\vec{g}$ (where $M = m_1 + m_2 + \cdots$ is the total mass of the object) has the same effect as does the combined effect of all the individual gravitational forces $m_i\vec{g}_i$. Let's proceed with the x coordinate (extension for the y coordinate is trivial), then

$$(m_1 g_1 + m_2 g_2 + \cdots)x_{cg} = m_1 g_1 x_1 + m_2 g_2 x_2 + \cdots$$

If we assume uniform g over the object (as is usually the case), then

$$x_{cg} = x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{11.4}$$

We see that the center of gravity is located at the center of mass as long as the object is in a uniform gravitational field. This is approximately true for everyday objects because $\vec{g}$ varies only a little along Earth's surface and decreases in magnitude only slightly with altitude.

Let's show that the net torque acting on a body can be written as if all the gravitation force is applied to the center of mass (when $\vec{g}$ is the same for all elements of a body).

We consider the individual elements of the body of mass M, and one of its elements, of mass m_i. A gravitational force acts on each such element and is equal to $\vec{F}_{gi} = m_i\vec{g}$. Each such force produces a torque on a single element about the origin $\vec{\tau}_i = \vec{r}_i \times m_i\vec{g}$. The net torque on all the elements of the body is then

$$\vec{\tau}_{net} = \sum \vec{\tau}_i = \sum \vec{r}_i \times m_i\vec{g} = \sum m_i\vec{r}_i \times \vec{g}$$

Next we divide and multiple this by the total mass of the body $M = \sum m_i$

$$\vec{\tau}_{net} = \frac{1}{M}\sum m_i\vec{r}_i \times M\vec{g}$$

Using the definition for the center of mass

$$\vec{r}_{cm} = \frac{1}{M}\sum m_i \vec{r}_i$$

we finally can write

$$\vec{\tau}_{net} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{F}_g. \quad (11.5)$$

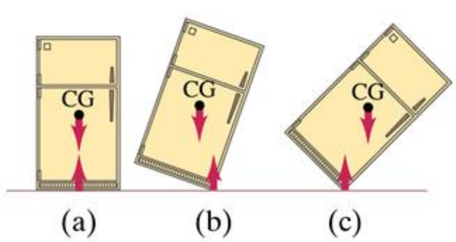
So, the total gravitational torque is the same as though the total gravitational force were acting on the position of the center of mass, which we also call the center of gravity.

Finding the center of gravity (or the center of mass) can be challenging for bodies of complex shapes. However, the center of gravity of a homogeneous sphere, cube, circular sheet, or rectangular plate is at its geometric center. Symmetry considerations can help with more complex shapes when we sometimes can locate the center of gravity by thinking of the body as being made of symmetrical pieces.

11.3 Few more words

A body in static equilibrium, if left undisturbed, will undergo no translational or rotational acceleration since the sum of all the forces and the sum of all the torques acting on it are zero. However, if the object is displaced slightly, three different outcomes are possible: (1) the object returns to its original position, in which case it is said to be in *stable equilibrium*; (2) the object moves even farther from its original position, in which case it is said to be in *unstable equilibrium*; or (3) the object remains in its new position, in which case it is said to be in *neutral equilibrium*.

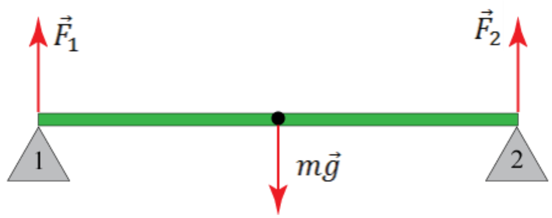
Quite often we are interested in maintaining stable equilibrium or *balance*, as we sometimes say. In general, an object whose center of gravity is below its point of support, such as a ball on a string, will be in stable equilibrium. If the center of gravity is above the base of support, we have a more complicated situation. Consider a standing refrigerator If it is tipped slightly, it will return to its original position due to the torque on it. But if it is tipped too far, it will fall over. The critical point is reached when the center of gravity is no longer above *the base of support*. In general, a body whose center of gravity is above its base of support will be stable if a vertical line projected downward from the center of gravity falls within the base of support. This is because the normal force upward on the object (which balances out gravity) can be exerted only within the area of contact, so that if the force of gravity acts beyond this area, a net torque will act to topple the object. Stability, then, can be relative. A brick lying on its widest face is more stable than a brick standing on its end, for it will take more of an effort to tip it over.



Standard physics textbooks usually offer two-dimensional problems in (x, y) plane. Then we have only three independent equations for equilibrium, namely two balance of forces equations and one balance of torques equations about a given rotational axis. However, in real life applications we often have more unknowns than equations. Consider also an unsymmetrically loaded car. What are the forces-all different on the four tires? Again, we cannot find them because we have only three independent equations with which to work. Similarly, we can solve an equilibrium problem for a table with three legs but not for one with four legs. Problems like these, in which there are more unknowns than equations, are called *indeterminate*. Yet solutions to indeterminate problems exist in the real world. What is eluding us in our efforts to find the individual forces by solving equations? The problem is that we have assumed-without making a great point of it that the bodies to which we apply the equations of static equilibrium are perfectly rigid. By this we mean that they do not deform when forces are applied to them. Strictly, there are no such bodies. The tires of the car, for example, deform easily under load until the car settles into a position of static equilibrium. To solve such indeterminate equilibrium problems, we must supplement equilibrium equations with some knowledge of elasticity, the branch of physics that describes how real bodies deform when forces are applied to them.

11.4 Statically undetermined systems

In this chapter we consider absolutely rigid bodies, no deformations under stress. However, such ideal picture is not always a realistic model. Let us consider a rigid beam on two supports and find forces on the beam from the supports.

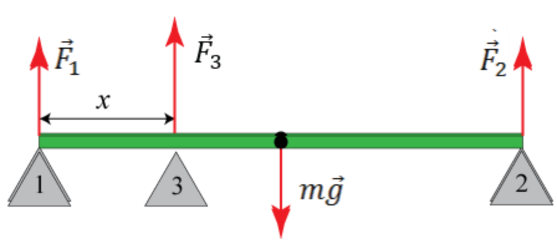


The conditions of equilibrium

$$F_1 + F_2 - mg = 0$$
$$F_2 L - \frac{mgL}{2} = 0$$

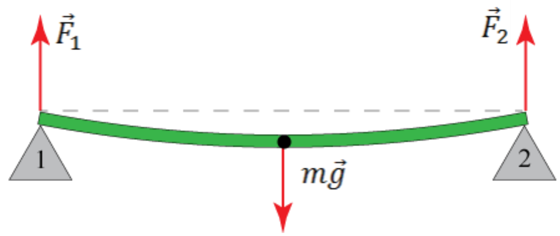
Where L is the length of the beam. Solving the system of equations gives $F_1 = F_2 = mg/2$. In this case the model of absolutely rigid body works well.

However, a problem about distribution of weight of a rigid beam between three supports cannot be solved since we still have two equations for the equilibrium but three unknowns.



$$F_1 + F_2 + F_3 - mg = 0$$
$$F_2 L + F_3 x - \frac{mgL}{2} = 0$$

Mechanical systems, like a beam on three supports, are *statically undetermined*. The same situation occurs in equilibrium of a rigid table on a horizontal surface. The problem can be solved for a three leg table but it is statically undetermined for four or more leg tables. Sure, a real beam and a real four leg table have a very specific distribution of their weights. The uncertainty comes from an approximation to consider objects (a beam, a table) as ideally rigid ones. A more accurate model for a beam should include deformation, or the beam bends under its own weight.



We need to take into account deformation of objects to find unique solutions.

11.5 Few guidelines for solving most common problems in "Equilibrium"

Three dimensional rigid body is a system with six "degrees of freedom" since it can move in three directions and rotate about three axis. In this chapter we consider two dimensional world, when a body is flat and located in xy −plane, and it can only rotate about z −axis. Then there are three degrees of freedom left, with three equations with conditions for static equilibrium in (x, y) plane.

$$F_{net,x} = 0 \qquad \text{(balance of forces along } x\text{)}$$
$$F_{net,y} = 0 \qquad \text{(balance of forces along } y\text{)}$$
$$\tau_{net,z} = 0 \qquad \text{(balance of torques around } z\text{).}$$

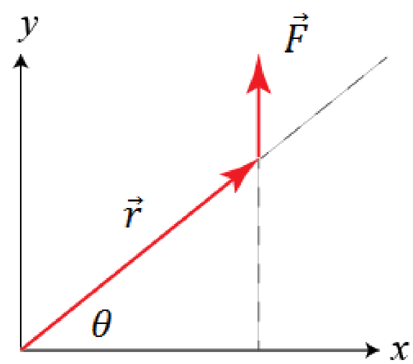
Problem solving:

1. You must always draw a free-body diagram showing all the forces acting on the selected body.
2. While choosing the reference point for calculating torques is completely arbitrary, once selected, the same point must be used to calculate all torques on a body. A wise choice can considerably simplify your calculations. For example, you can reduce the number of unknowns by choosing the axis so that one of the unknown forces passes through the axis; then this force will produce zero torque and so will not appear in the equation.
3. Remember that torque has a direction (positive for counterclockwise "rotation" and negative for clockwise one).
4. You always need as many equations as you have unknowns. For more complicated problems you may need to computer torques with respect to two or more axes to obtain enough equations.
5. Using the component form is much more efficient way to calculate torques.

There is a group of problems in static equilibrium that is called as "balance" problems. Examples of such problems include: two children on a seesaw, a person walking along a plank extended beyond a support area. Quite often such problems can be easily solved by calculating the position of the center of gravity instead of dealing with forces and torques. Remember that in general, a body whose center of gravity is above its base of support will be stable if a vertical line projected downward from the center of gravity falls within the base of support. The critical point is reached when the center of gravity is no longer above the base of support.

A quick note on calculating torque (again)

$$\vec{\tau} = \vec{r} \times \vec{F}$$



1. Using angles

$$\tau = rF \sin\theta = rF_{\perp} = r_{\perp}F$$

As you probably see using angles may need more steps. Namely the angle between vectors (counting from $\vec{r}$ to $\vec{F}$) on this diagram is $90^0 - \theta$. Then

$$\tau = rF \sin\theta = rF \sin(90^0 - \theta)$$

Since $\sin(90^0 - \theta) = \cos\theta$

$$\tau = rF \sin(90^0 - \theta) = r\cos\theta\, F = xF$$

2. Using components in xy −plane we have

$$\tau_z = r_xF_y - r_yF_x.$$

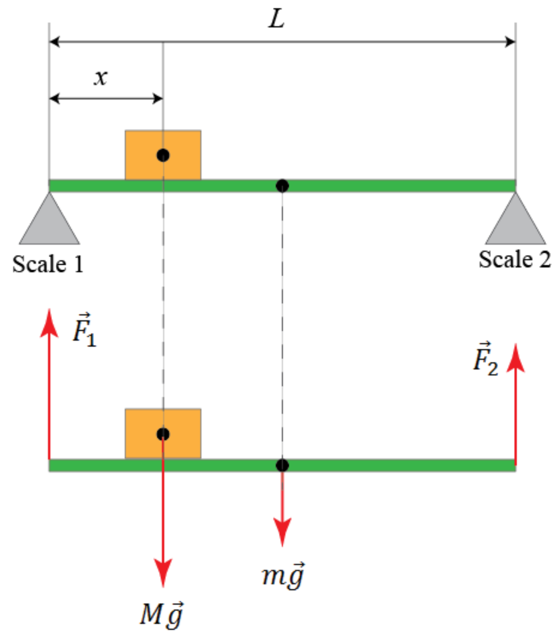
This way is much more straightforward. For the diagram you immediately get $\tau_z = xF$.

The component form is invaluable for arbitrary orientations of vectors $\vec{r}$ and $\vec{F}$ relative to the coordinate system. It also provides you with a proper sign (positive or negative) for torques.

11.6 Examples

Example 11-1

A uniform beam, of length L and mass m is at rest with its ends on two scales. A uniform block, with mass M, is at rest on the beam, with its center a distance x from the beam's left end. What do the scales read?



SOLUTION:

1. Physics – static equilibrium

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

3. We choose the beam as the object of interest (choosing the block does not help to find the forces on the scales). The free-body diagram shows all forces acting on the beam.

The first condition of equilibrium (balance of forces) for the y-direction (there are no forces acting along x-direction and $F_{net,x}$ provides no information)

$$F_{net,y} = F_1 + F_2 - Mg - mg = 0$$

For the second condition we need to choose a rotational axis perpendicular to the plane xy. Let's choose it through the left end of the beam. Using the component form for calculating torques we get

$$\tau_z = 0F_1 - xMg - \frac{L}{2}mg + LF_2 = 0$$

4. Now we have a system of two equations with two unknowns

$$F_1 + F_2 - Mg - mg = 0$$
$$-xMg - \frac{L}{2}mg + LF_2 = 0$$

Since the second equation (balance of torques) has only one unknown we can solve it immediately

$$F_2 = \frac{x}{L}Mg + \frac{1}{2}mg$$

With this solution we can solve the first equation to find the first force

$$F_1 = -F_2 + Mg + mg = -\frac{x}{L}Mg - \frac{1}{2}mg + Mg + mg = \left(1 - \frac{x}{L}\right)Mg + \frac{1}{2}mg$$

5. Calculations – no calculation for this problem

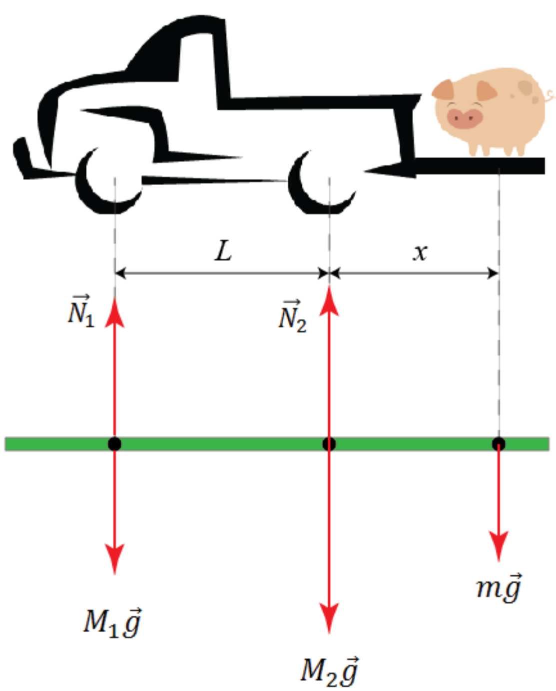
6. Looking back – we cannot have a dimension analysis here, but we can consider a special case when we may guess right answer. It is clear that placing the block in the middle would give equal reading for the scales. Let's see our solutions for $x = L/2$

$$F_1 = \left(1 - \frac{1}{2}\right)Mg + \frac{1}{2}mg = \frac{1}{2}Mg + \frac{1}{2}mg, \quad F_2 = \frac{1}{2}Mg + \frac{1}{2}mg.$$

<u>Example 11-2</u>

A pickup truck has a wheelbase of L meters. Ordinary M_1 kg rests on the front wheels, and M_2 kg on the rear wheels, when the truck is parked on a level road. A pig of m kg is now placed on the tailgate, x meters behind the rear axel.

a) How much total weight now rests on the front wheels? On the rear wheels?
b) How much weight would need to be placed on the tailgate to make the front wheels come off the ground?



SOLUTION:

1. Physics – static equilibrium

Basically this problem is not much different from the first example

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

3. We choose the truck as the object of interest (choosing the pig does not help to find the normal forces on the wheels). The free-body diagram shows all forces acting on the truck

The first condition of equilibrium (balance of forces) for the y-direction (there are no forces acting along x-direction and $F_{net,x}$ provides no information)

$$F_{net,y} = N_1 + N_2 - M_1g - M_2g - mg = 0$$

For the second condition we need to choose a rotational axis perpendicular to the plane xy. Let's choose it through the left wheels. Using the component form for calculating torques we get

$$\tau_z = 0N_1 - 0M_1g + LN_2 - LM_2g - (L + x)mg = 0$$

The condition for the front wheels to come off the ground means

$$N_1 = 0$$

Later we will use this condition when we have a solution for N_1

4. Now we have a system of two equations with two unknowns

$$N_1 + N_2 - M_1 g - M_2 g - mg = 0$$
$$LN_2 - LM_2 g - (L + x)mg = 0$$

Since the second equation (balance of torques) has only one unknown we can solve it immediately

$$N_2 = M_2 g + \left(1 + \frac{x}{L}\right) mg$$

With this solution we can solve the first equation to find the first force

$$N_1 = -N_2 + M_1 g + M_2 g + mg = -M_2 g - \left(1 + \frac{x}{L}\right) mg + M_1 g + M_2 g + mg = M_1 g - \frac{x}{L} mg$$

from $N_1 = 0$

$$N_1 = M_1 g - \frac{x}{L} mg = 0$$

follows

$$m = M_1 \frac{L}{x}$$

With so heavy pig the front wheels will come off the ground. Let's note that the large x the fewer load needed on the tailgate to have the front wheels off the ground.

5. Calculations – no calculation for this problem

6. Looking back – we cannot have a dimension analysis here, but we can consider a special case when we may guess right answer. It is clear that without the pig and for $M_1 = M_2 = M$ we should expect the same normal force acting on both front and rear wheels.

$$N_1 = M_1 g = Mg, \quad N_2 = M_2 g = Mg.$$

We got it right.

Method II for question b).

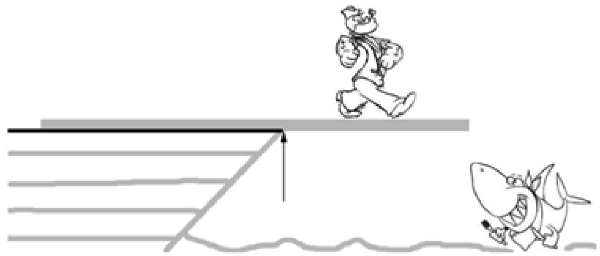
By the way, the second question is an equilibrium question. It can be easily answered by calculating the position for the center of gravity. Let's choose the position of the rear wheels as an origin. Then the center of gravity of the systems is

$$x_{cg} = \frac{-M_1 L + M_2 0 + mx}{M_1 + M_2 + m}$$

The critical point is reached when the center of gravity is just above the rear wheels or $x_{cg} = 0$. Then we have $M_1 L = mx$, or $m = M_1 L/x$, or this the same result that we got using forces and torques.

Example 11-3

A long uniform wooden plank of length L=9.0 m and mass of M=100.0 kg rests on a deck of a ship. Only two thirds of the plank is on the deck. How far from the edge of the ship can Popeye of mass m=60 kg go on the hanging side of the plank if the plank is to remain at rest?



Assume that the plank does not slide back as Popeye goes forward

SOLUTION:

1. Physics – static equilibrium, balance problem

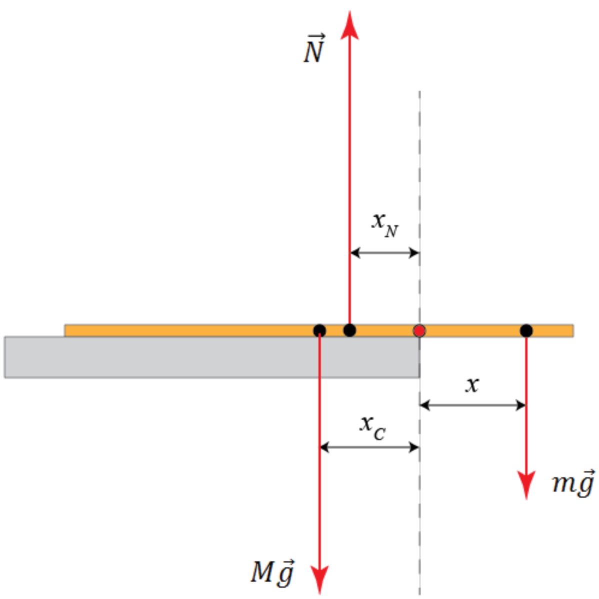
2. The basic equations for equilibrium

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

For balance problems we calculate the center of gravity

$$x_{cg} = x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

3. We choose the wooden plank as the object of interest. The free-body diagram shows all forces acting on the plank



Method I: using the center of gravity. Choosing the red dot as the origin ($x = 0$) we have

$$x_{cg} = \frac{-Mx_c + mx}{M + m}$$

The critical point is reached when the center of gravity is just above the edge of the ship (the red dot). Then from $x_{cg} = 0$ one has

$$x = x_C\frac{M}{m}$$

Method II: general approach to equilibrium problems. The problem looks very similar the first two problems. However, there is something new. Where is the normal force applied? Should it be at the same x position as the gravity on the plank? If we place the normal vector at the location of the center of gravity, then we would get rotation from $m\vec{g}$ around the center of gravity!

One may say, well I expect the normal force be at the edge of the ship when Popeye is going to start falling.

Let's avoid any guesses and proceed with equations.

The first condition of equilibrium (balance of forces) for the y-direction (there are no forces acting along x-direction and $F_{net,x}$ provides no information).

$$F_{net,y} = N - Mg - mg = 0$$

For the second condition we need to choose a rotational axis perpendicular to the plane xy. Let's choose it through the "red" point (the edge of the ship) as the origin for the coordinate system. Using the component form for calculating torques we get

$$\tau_z = x_C Mg - x_N N - xmg = 0$$

4. Now we have a system of two equations with two unknowns (N and x_N) for some position x.

$$N - Mg - mg = 0$$
$$x_C Mg - x_N N - xmg = 0$$

Solving for the unknowns

$$N = Mg + mg$$

$$x_N = \frac{x_C Mg - xmg}{(M+m)g} = \frac{X_C M - xm}{M+m}$$

As we can see the location of the normal force is not at the center of gravity!

Now, when the position of the normal force at the edge of the ship, then this is where a delicate balance happens, any more step and Popeye goes swimming (normal force cannot act from the air, it has to be from a surface). Thus, the balance condition is $x_N = 0$. From this condition

$$x_N = \frac{x_C M - xm}{M+m} = 0$$

we get

$$x = x_C \frac{M}{m}$$

5. Calculations

Note that from the condition for the plan's position follows that $x_C = (1/2 - 1/3)L = L/6$

$$x = \frac{1}{6} 9\,m \frac{100\,kg}{60\,kg} = 2.5\,m$$

6. Looking back – correct dimensions.

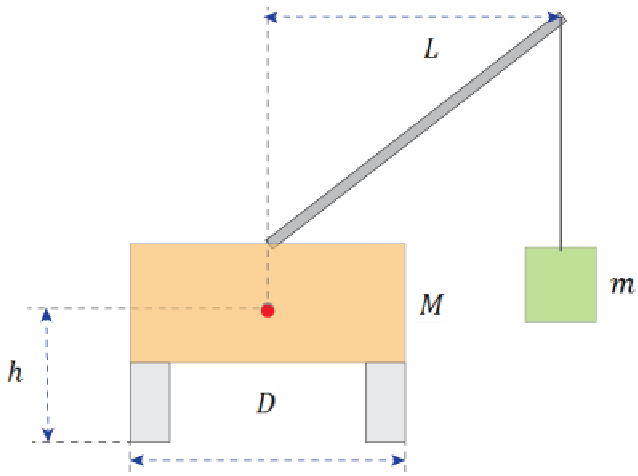
Let's consider a special case when we may guess right answers. If our solution is able to reproduce our special case consideration then we probably have a correct solution.

Without Popeye the position of the normal force should be at the same horizontal position as the gravitational force. For $m = 0$ we have $x_N = x_C$. Correct.

By the way, using the "center-of-gravity" approach (method I) to this balance problem produced the answer in just few equations.

Example 11-4

A crane is mounted on a truck. The mass of the truck is $M = 5{,}000\ kg$, the width of the truck (the distance between outer sides of the tires) is $D = 3.2\ m$, the center of mass of the truck (with the crane) is $h = 2\ m$ above the ground, the end of the boom of the crane is extended $L = 5\ m$ from the center of the gravity of the truck (assume that the mass of the boom is much less than the mass of the truck). What is the maximum mass of a load the crane can lift without going overturned?



SOLUTION:

1. Physics – static equilibrium, balance problem

2. The basic equations for equilibrium

$$\begin{aligned} F_{net,x} &= 0 \qquad (balance\ of\ forces)\\ F_{net,y} &= 0 \qquad (balance\ of\ forces)\\ \tau_{net,z} &= 0 \qquad (balance\ of\ torques). \end{aligned}$$

the center of gravity

$$x_{cg} = x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

3. While this problem can be solved using the equilibrium conditions, it is much easier to approach it as a balance problem. Choosing the origin ($x = 0$) at the right side of the truck we have

$$x_{cg} = \frac{-\frac{Md}{2} + m\left(L - \frac{D}{2}\right)}{M + m}$$

4. The critical point is reached when the center of gravity is just above the right side of the truck, then from $x_{cg} = 0$

$$m = M\frac{D}{2L - D}$$

5. Calculations

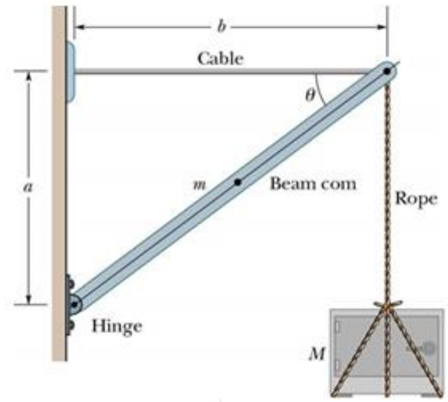
$$m = 5000\ kg\ \frac{3.2\ m}{2 \cdot 5\ m - 3.2\ m} = 2352\ kg$$

6. Looking back – we have right dimensions, and the results looks as a credible one.

Example 11-5

A safe of mass M hanging by a rope from a boom with dimensions a and b. The boom consists of a hinged beam and a horizontal cable that connects the beam to a wall. The uniform beam has a mass m. The masses of the cable and the rope are negligible.

a) What is the tension in the cable?
b) Find the magnitude of the net force on the beam from the hinge.



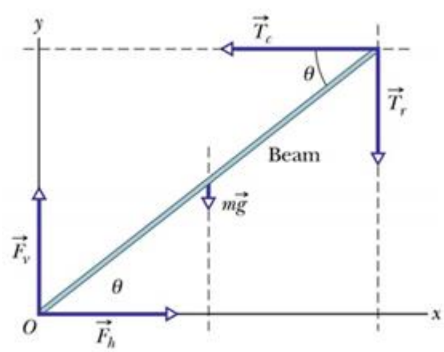
SOLUTION:

1. Physics – static equilibrium

This problem, unlike the first two, is a two-dimensional problem

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$



3. We choose the beam as the object of interest (see why not the cable or the rope?)

Let's note that $T_r = Mg$ from the equilibrium condition for the safe. We will use it in a moment.

The balance of forces

$$F_{net,x} = F_h - T_c = 0$$
$$F_{net,y} = F_v - mg - Mg = 0$$

For the balance of torques we need to choose a rotational axis perpendicular to the plane *xy*. Let's choose it through the hinge. Thus we will have fewer unknowns in our equation.

Using the component form for calculating torques we get

$$\tau_z = -\frac{b}{2}mg - bMg + aT_c = 0$$

Using the component from makes it much easier to calculate the torques. Otherwise you would be wrestling with the angles.

4. Now we have a system of three equations with three unknowns (T_c, F_h, F_v)

$$F_h - T_c = 0$$
$$F_v - mg - Mg = 0$$

$$-\frac{b}{2}mg - bMg + aT_c = 0$$

The last equation immediately gives the tension in the cable

$$T_c = \frac{b}{2a}mg + \frac{b}{a}Mg = \frac{b}{a}\left(\frac{1}{2}m + M\right)g$$

Then the horizontal and vertical components of the force on the hinge

$$F_h = T_c = \frac{b}{a}\left(\frac{1}{2}m + M\right)g$$
$$F_v = (m + M)g$$

The magnitude of the net force on the beam from the hinge

$$F_{hinge} = \sqrt{F_h^2 + F_v^2}$$

5. Calculations – no calculation for this problem

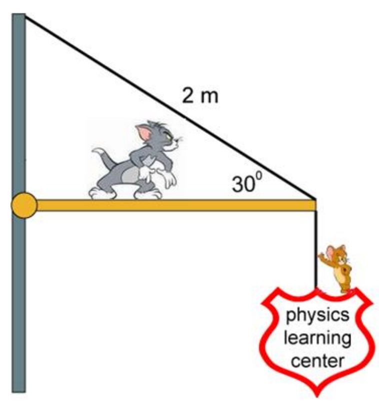
6. Looking back – we cannot have a dimension analysis here, but we can consider a special case when we may guess right answer.

We can assume that for a vertical beam there is no need for the supporting cable, or if $b = 0$, then $T_c = 0$. This is exactly what follows from our general solution

<u>Example 11-6</u>

A sign of mass M_s= 8.0 kg is supported by a uniform m=4.0 kg beam as shown on the figure. A cat (Tom) of mass M_c=5.0 kg walks slowly from the wall toward the end of the beam to talk to Jerry (m_m= 0.2 kg) probably about physics of equilibrium.

a) Will the cable break before the cat reaches the end of the beam, if the cable will break up under tension of 280.0 N?
b) If so, how far from the wall the cat will be when the cable breaks?



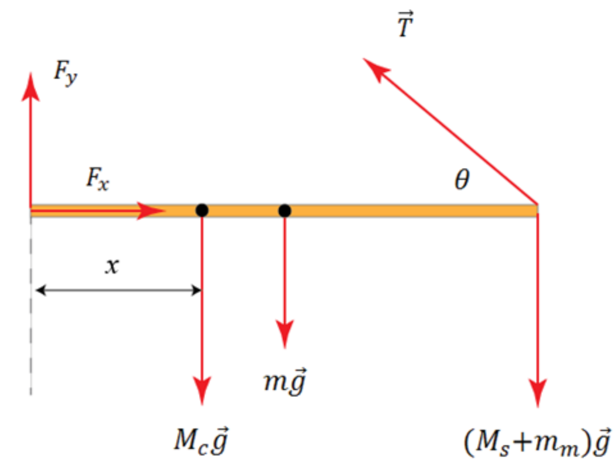
The cable is d=2.0 meters long. It makes 30^0 degrees with the beam. Assume that the mass of the cable is negligible.

SOLUTION:

1. Physics – static equilibrium

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$



3. We choose the beam as the object of interest (not the cat, or the mouse or the cable). The beam is $L = d\cos\theta$ long.

There are two ways to approach the problem.

a. We can place the cat at the end of the beam (distance L from the wall) to see if the cable breaks under this condition. If it breaks then we have to solve the problem again to find at what position x of the cat it will happen.

b. Or we can find the tension in the cable having the cat distance x from the wall. Then we use the condition $T = T_{critical}$ to find $x_{critical}$. If it less than the size of the beam, then the cable will break. The second way is preferable since we have to solve the problem once.

The balance of forces

$$F_{net,x} = F_x - T\cos\theta = 0$$
$$F_{net,y} = F_y - M_c g - mg - (M_s + m_m)T\sin\theta = 0$$

Let's choose the rotational axis at the left end of the beam (thus we will have fewer unknowns in our equation. Using components the balance of torques is

$$\tau_z = -xM_c g - \frac{L}{2}mg - (M_s + m_m)g + LT\sin\theta = 0$$

(Note that we used here the component equation for torques, namely $\tau_z = r_x F_y - r_y F_x$.)

4. We have a system of three equations with three unknowns (x, F_x, F_y), but we need to find only one – the distance when the cable will break.

$$F_x - T\cos\theta = 0$$
$$F_y - M_c g - mg - (M_s + m_m)T\sin\theta = 0$$
$$-xM_c g - \frac{L}{2}mg - L(M_s + m_m)g + LT\sin\theta = 0$$

With the wise choice of the rotational axis, we have it right in the third equation where we use $T = T_{critical}$.

$$x = L\frac{1}{M_c}\left(\frac{T_{critical}\sin\theta}{g} - \frac{1}{2}m - M_s - m_m\right)$$

5. Calculations are straightforward

For the given critical tension, the cat should be 1.42 m from the wall to have the cable broken. However, the beam is $L = 2.0\ m\ \cdot \cos 30^0 = 1.73\ m$. That means that the cable will break before Tom gets to the end of the beam.

6. Looking back.

Since tension is measured in [N] (where [N]=[m][g]), then we have right units for the distance.

A quick analysis of equation for x shows that for heavier cats the critical distance is getting shorter.

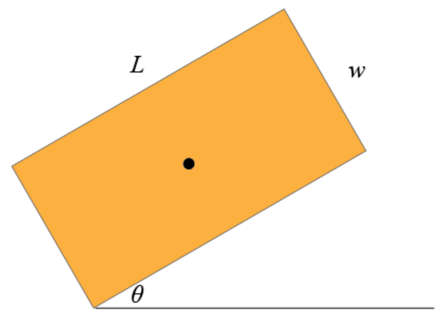
If there is no Tom and Jerry ($M_c = 0, m_m = 0$), the tension in the cable

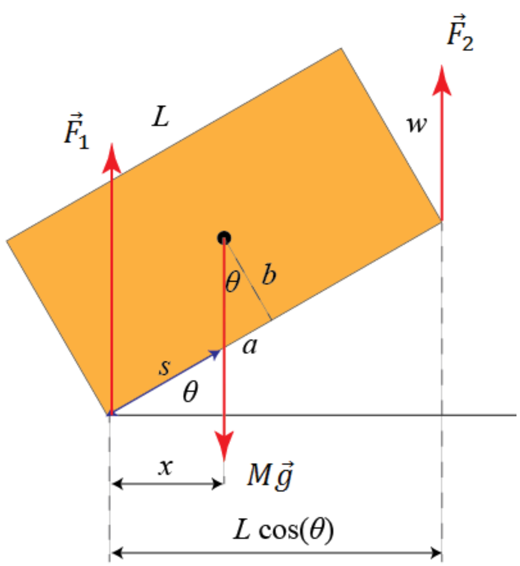
$$T = \frac{1}{\sin\theta}\left(\frac{1}{2}m + M_s\right)g$$

<u>Example 11-7</u>

This problem may look a bit challenging but could be a practical one.

Two friends are carrying a 200-kg crate up a flight of stairs. The crate is L=1.25 m long and w=0.500 m high, and its center of gravity is at its center. The stairs make a 45^0 angle with respect to the floor. The crate is also carried at a 45^0 angle, so its bottom side is parallel to the slope of the stairs. If the force each person applies is vertical, what is the magnitude of each of these forces? Is it better to be the person above or below on the stairs?





SOLUTION:

1. Physics – static equilibrium.

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

3. We choose the crate as the object of interest. Here θ is the angle that the crate makes with the floor (not with the flight of stairs. Since this problem involves using geometry and trigonometry we better have a good set of notations on our diagram.

The balance of forces

$$F_{net,x} = 0$$
$$F_{net,y} = F_1 + F_2 - Mg = 0$$

Let's choose the rotational axis at the lower left end of the crate. Using components ($\tau_z = r_x F_y - r_y F_x$)
the balance of torques is

$$\tau_z = -xMg + L\cos\theta\, F_2 = 0$$

4. We have a system of two equations with two unknowns (F_1, F_2).

$$F_1 + F_2 - Mg = 0$$
$$-xMg + L\cos\theta\, F_2 = 0$$

Now we need to find x in terms of L, w, θ. Here geometry + trigonometry come. For the small "orange" triangle inside the crate we have the same angle θ as the angle with the floor (do you see why?). Since we know $b = w/2$ we can find a

$$a = b\tan\theta = \frac{w}{2}\tan\theta$$

Since $s + a = L/2$

$$s = \frac{L}{2} - \frac{w}{2}\tan\theta$$

because $x = s\cos\theta$ we finally get

$$x = \left(\frac{L}{2} - \frac{w}{2}\tan\theta\right)\cos\theta = \frac{1}{2}(L\cos\theta - w\sin\theta)$$

With this x we can write the solution from the balance of torques as

$$F_2 = \frac{xMg}{L\cos\theta} = \frac{1}{2}\left(\frac{L\cos\theta}{L\cos\theta} - \frac{w\sin\theta}{L\cos\theta}\right)Mg = \frac{1}{2}\left(1 - \frac{w}{L}\tan\theta\right)Mg$$

Then the first force can be found from the balance of forces

$$F_1 = Mg - F_2 = \frac{1}{2}\left(1 + \frac{w}{L}\tan\theta\right)Mg$$

5. Calculations

$$F_1 = \frac{1}{2}\left(1 + \frac{0.50\ m}{1.25\ mL}\tan 30^0\right)200\ kg\ \cdot 9.8\ m/s^2 = 1372\ N$$
$$F_2 = \frac{1}{2}\left(1 - \frac{0.50\ m}{1.25\ mL}\tan 30^0\right)200\ kg\ \cdot 9.8\ m/s^2 = 588\ N$$

It is clear that it is better to be the person above on the stairs. But if you are a good friend you will probably take the place at the lower end.

6. Looking back.

Let's consider a special case (or two) when we may guess right answers. If our general solution is able to reproduce our special case considerations then probably we have a correct general solution.

It is clear that if the friend were on the floor (then the angle $\theta = 0$) we should expect equal load for both. From our general solution it follows for $\theta = 0$

$$F_1 = \frac{1}{2}Mg, \quad F_2 = \frac{1}{2}Mg$$

What if it was not a crate but a beam (or a log) with negligible width comparing to its length ($w \ll L$)? Then we have

$$F_1 = \frac{1}{2}Mg, \quad F_2 = \frac{1}{2}Mg$$

That means that we see the difference if you are on a flight of stairs carrying a wide object. By the way we can easily find such an angle when the person above on the stairs does not carry anything, or $F_2 = 0$, thus from

$$F_2 = \frac{1}{2}\left(1 - \frac{w}{L}\tan\theta\right)Mg = 0$$

we have

$$\tan\theta = \frac{L}{w}$$

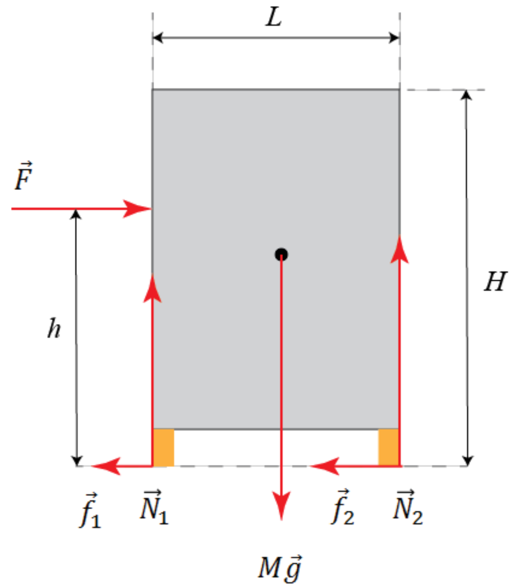
(for the given crate this angle is about 68^0).

<u>Example 11-8</u>

You are trying to move a dresser of mass M and dimensions of length L and height H by pushing it with a constant velocity by applying a horizontal force F a height h above the floor. The coefficient of kinetic friction between the dresser and the floor is μ. The ground exerts upward normal forces of magnitudes N_1 and N_2 at the two ends of the dresser.

Find the height about the floor that you can push a dresser before it starts to topple.

SOLUTION:



1. Physics – static equilibrium, friction

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

Friction

$$f = \mu N$$

3. The balance of forces

$$F_{net,x} = F - f_1 - f_2 = F - \mu N_1 - \mu N_2 = 0$$
$$F_{net,y} = N_1 + N_2 - Mg = 0$$

Let's choose the rotational axis at the RIGHT end of the dresser. Using components ($\tau_z = r_x F_y - r_y F_x$) the balance of torques is

$$\tau_z = -hF - LN_1 + \frac{L}{2}Mg = 0$$

4. Our system of equations.

$$F - \mu N_1 - \mu N_2 = 0$$
$$N_1 + N_2 - Mg = 0$$
$$-hF - LN_1 + \frac{L}{2}Mg = 0$$

From the first equation $F = \mu(N_1 + N_2)$. Since $N_1 + N_2 = Mg$ (from the second equation), the we have that the force to push with constant velocity is

$$F = \mu Mg$$

or the net frictional force. With this information, the last equation immediately gives N_1

$$N_1 = \frac{1}{2}Mg - \frac{h}{L}\mu Mg$$

and then

$$N_2 = Mg - N_1 = \frac{1}{2}Mg + \frac{h}{L}\mu Mg$$

Let's note that the left end will lose contact with the floor ($N_1 = 0$) at the moment the dresser begins to topple. Thus

$$N_1 = 0 = \frac{1}{2}Mg - \frac{h_m}{L}\mu Mg$$

where h_m is the maximum height about the floor that you can push a dresser before it starts to topple.

$$h_m = \frac{L}{2\mu}$$

5. No calculations

6. Looking back.

Let's consider a special case (or two) when we may guess right answers. If our general solution is able to reproduce our special case considerations then probably we have a correct solution.

It is clear that in absence of friction $\mu = 0 \; and \; h_m \to \infty$ there is no way to get a problem with furniture (but then how could we move furniture if we do not have traction?).

Example 11-9

A car is travelling at a constant speed v on a horizontal curved ramp of radius $R = 100 \; m$. Assume the height of the car's center of mass above the ground is h = 1.0 m, and the separation between its wheels (the axle track or car's width) is $L = 1.6 \; m$. The maximum coefficient of static friction between the tires and the surface of the road is $\mu_s = 0.9$.

a) Find its maximum speed if the car is to negotiate this path without skidding.
b) Find its maximum speed if the car is to negotiate the path without overturning.

SOLUTION:

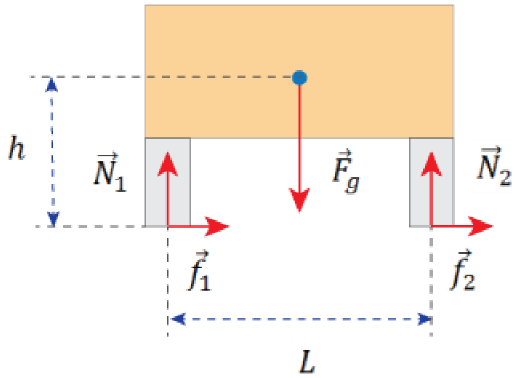
1. Physics – circular motion with acceleration, friction, rotational equilibrium.

Let's note that the first question has been answered in example 5-11 (question a), namely, $v_{max} = \sqrt{\mu_s g R}$. However, it is interesting to compare the two solutions, for skidding and overturning, or what is going to happen first. Now, we concentrate on question b).

2. The basic equations

$$F_{net,x} = \frac{Mv^2}{R} \qquad (centripetal \; force)$$
$$F_{net,y} = 0 \qquad (balance \; of \; forces)$$
$$\tau_{net,z} = 0 \qquad (balance \; of \; torques).$$

Friction $\;f = \mu N$



3. The balance of forces is a bit more complicated because the car is following a circular path (with a centripetal force to provide centripetal acceleration in x −direction)

$$F_{net,x} = f_1 + f_2 = \frac{Mv^2}{R}$$
$$F_{net,y} = N_1 + N_2 - Mg = 0$$

where $f_1 = \mu N_1$ and $f_2 = \mu N_2$.

Let's choose the rotational axis at the center of the gravity (center of mass) of the car. Using components for the torques we get the balance equation as

$$\tau_z = -N_1 \frac{L}{2} + f_1 h + N_2 \frac{L}{2} + f_2 h = 0$$

or

$$\tau_z = -N_1 \frac{L}{2} + \mu N_1 h + N_2 \frac{L}{2} + \mu N_2 h = 0$$

4. Now we have three equations with three unknowns, explicitly N_1, N_2 and μ. One may be surprised that we have the coefficient μ as unknown. Indeed, it is less than the maximum static friction coefficient μ_s if one drives with speed less than $v_{max} = \sqrt{\mu_s g R}$. So, let's find all the unknowns. Using the second equation for balance of forces together with balance of torques gives

$$N_1 + N_2 = Mg$$
$$N_1 \left(\frac{L}{2} - \mu h\right) = N_2 \left(\frac{L}{2} + \mu h\right)$$

then solving the system of equation

$$N_1 = Mg \left(\frac{\frac{L}{2} + \mu h}{L}\right), \quad N_2 = Mg \left(\frac{\frac{L}{2} - \mu h}{L}\right)$$

It is interesting that the two normal forces are different if moving on a circular pass.

However, for a car moving along a straight line (no radial motion, or $\mu = 0$) then $N_1 = N_2 = Mg/2$.

The car is about to overturn if $N_2 = 0$ that happens if

$$\frac{L}{2} - \mu h = 0$$

or $\mu = L/2h$.

From the first balance equation

$$f_1 + f_2 = \mu N_1 + \mu N_2 = \mu(N_1 + N_2) = \mu Mg = \frac{Mv^2}{R}$$

or

$$\mu = \frac{v^2}{gR}$$

Using $\mu = L/2h$ from the above solution gives

$$\frac{L}{2h} = \frac{v^2}{gR}$$

$$v_{over} = \sqrt{\frac{L}{2h} gR}$$

where v_{over} is the maximum speed a car may have before starting to overturn. By the way, the larger h the lower the critical v_{over} speed (that is the case for SUVs where the ratio $L/2h$ is lower than for regular cars).

5. For the given conditions

$$v_{max} = \sqrt{\mu_s gR} = \sqrt{0.9 \cdot 9.8\frac{m}{s^2} \cdot 100m} = 30\ m/s = 67\ mph$$

$$v_{over} = \sqrt{\frac{L}{2h} gR} = \sqrt{\frac{1.6\ m}{2 \cdot 1\ m} 9.8\frac{m}{s^2} \cdot 100m} = 28\ m/s = 63\ mph$$

6. Looking back.

For the given conditions the care will rather overturn than skid.

<u>Example 11-10</u>

A driver of a car on a horizontal road makes an emergency stop by applying the brakes so that all four wheels lock and skid along the road. The coefficient of kinetic friction between tires and road is $\mu = 0.80$. The wheelbase (the separation between the front and rear axles) is $D = 2.8\ m$, and the center of mass of the car is located at distance $d = 1.2\ m$ behind the front axle and distance $h = 0.90\ m$ above the road. The car mass is $M = 1000\ kg$. Find the magnitude of the normal force on rear wheels and the normal force on front wheels.

SOLUTION:

1. Physics – motion with translational acceleration, friction, rotational equilibrium. (While the car is not in translational equilibrium, it is in rotational equilibrium!).

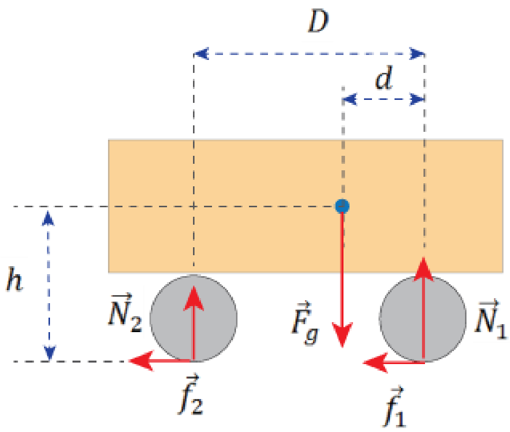
This problem is remarkably similar to example 11-8.

2. The basic equations

$$F_{net,x} = Ma \qquad (acceleration\ in\ x\ direction)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$

$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

Friction $f = \mu N$



3. For translational forces

$$F_{net,x} = -f_1 - f_2 = -Ma$$
$$F_{net,y} = N_1 + N_2 - Mg = 0$$

where $f_1 = \mu N_1$ and $f_2 = \mu N_2$.

Let's choose the rotational axis at the center of the gravity (center of mass) of the car. Then

$$\tau_z = N_1 d - f_1 h - N_2(D-d) - f_2 h = 0$$

or

$$\tau_z = N_1 d - \mu N_1 h - N_2(D-d) - \mu N_2 h = 0$$

4. Three equations but two unknowns, namely N_1, N_2. Is there a problem? In fact the first two equations can be considered as one because

$$F_{net,x} = -f_1 - f_2 = -Ma \qquad \mu(N_1 + N_2) = Ma$$

$$F_{net,y} = N_1 + N_2 - Mg = 0 \qquad N_1 + N_2 = Mg$$

they are connected in a simple way $a = \mu g$.

Then we solve simultaneously these two equations

$$N_1 + N_2 = Mg$$

$$N_1(d - \mu h) = N_2(D - d + \mu h) = 0$$

Solving the system of equation gives

$$N_1 = Mg\left(\frac{D - d + \mu h}{D}\right), \quad N_2 = Mg\left(\frac{d - \mu h}{D}\right)$$

where $\mu = a/g$

5. For the given conditions

$$N_1 = Mg\left(\frac{D - d + \mu h}{D}\right) = 8120\ N$$

$$N_2 = Mg\left(\frac{d - \mu h}{D}\right) = 1680\ N$$

6. Looking back.

Much more force is on the front wheels. Let's see what we get if there was no acceleration $\mu = 0$ and the center of gravity was in the middle $d = D/2$. Then we have $N_1 = N_2 = Mg/2$.

Example 11-11

A ladder having a uniform density and a mass m = 10 kg rests against a frictionless vertical wall at an angle of 60⁰. The lower end rests on a flat surface where the coefficient of static friction is 0.40. A student with a mass M = 65 kg attempts to climb the ladder. What fraction of the length L = 5 m of the ladder will the student have reached when the ladder begins to slip?

SOLUTION:

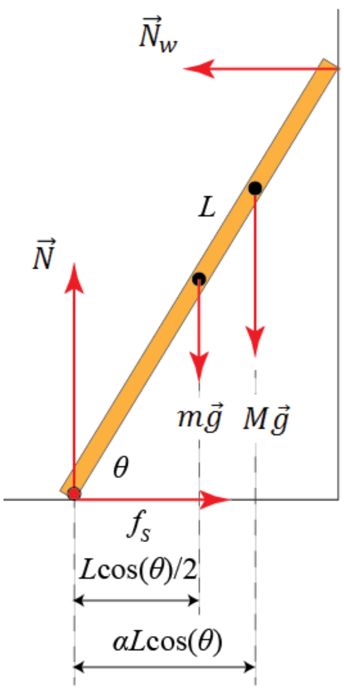
1. Physics – static equilibrium, friction

2. The basic equations

$$F_{net,x} = 0 \qquad (balance\ of\ forces)$$
$$F_{net,y} = 0 \qquad (balance\ of\ forces)$$
$$\tau_{net,z} = 0 \qquad (balance\ of\ torques).$$

Friction

$$f = \mu N$$



3. The balance of forces

$$F_{net,x} = f_s - N_w = 0$$
$$F_{net,y} = N - Mg - mg = 0$$

Let's choose the rotational axis at the low end of the ladder (the red dot). Also we assume that α is a fraction of the leader that the student has reached, or he climbed αL along the ladder.

Using components $(\tau_z = r_x F_y - r_y F_x)$ the balance of torques is

$$\tau_z = -\frac{L}{2} mg\cos\theta - \alpha LMg\cos\theta + LN_w\sin\theta = 0$$

4. After using $f_s = \mu N$ our system of equations is

$$\mu N - N_w = 0$$
$$N - Mg - mg = 0$$
$$-\frac{L}{2}\cos\theta\, mg - \alpha L\cos\theta\, Mg + L\sin\theta\, N_w = 0$$

There are three unknowns in the system: N, N_w and α. From the second equation

$$N = (m + M)g$$

then from the first equation

$$N_w = \mu(m + M)g$$

and from the third equation

$$\alpha = \frac{L\sin\theta\, N_w - -\frac{L}{2}\cos\theta\, mg}{L\cos\theta\, Mg} = \frac{L\sin\theta\,\mu(m+M)g - -\frac{L}{2}\cos\theta\, mg}{L\cos\theta\, Mg}$$

and after some algebra

$$\alpha = \mu\left(\frac{m+M}{M}\right)\tan\theta - \frac{1}{2}\left(\frac{m}{M}\right)$$

5. Calculations

$$\alpha = 0.4\left(\frac{10\,kg + 65\,kg}{65\,kg}\right)\tan 60^0 - \frac{1}{2}\left(\frac{10\,kg}{65\,kg}\right) = 0.72$$

or the student will beat $h = 0.72 \cdot 5\,m \cdot \sin 60^0 \; = 3.1\,m$ above the ground.

6. Looking back.

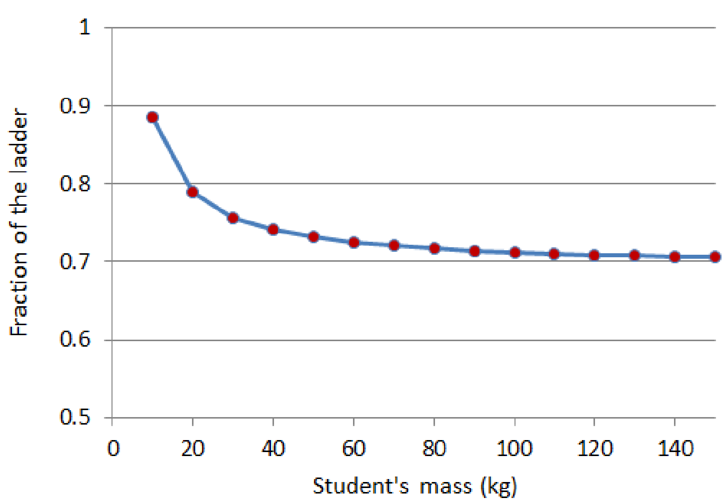
Units (actually no units) are correct.

I cannot imagine some special case with an obvious solution.

The ladder problem is a classic example in most textbooks. You might noticed that we took into account only one frictional force, namely from the floor. Most discussions of this problem assume that the static frictional force between the ladder and wall can be ignored. Can it? Indeed, we would expect one. However in this case the problem is getting very complicated with no unique solution (four unknowns but three equations!). Without modeling the elasticity of the ladder, it is not possible to solve the ladder problem for all the external forces. It is interesting how a simple problem can get complicated in no time, just by adding one more force

More for the ladder problem: We can analyze the major dependences.

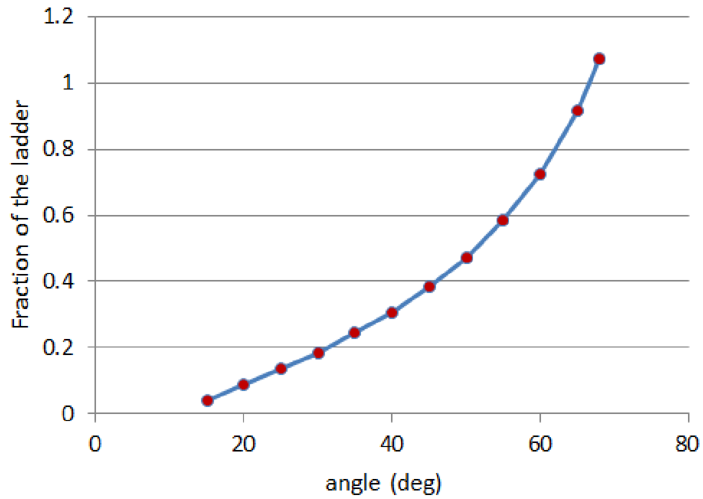
a) let's see how α changes with M for fixed $\theta = 60^0$, $m = 10\,kg$ and $\mu = 0.4$



This is quite interesting. As the mass increases the fraction decreases but only to a certain critical point on the ladder.

b) let's explore the angle dependence for $M = 65\ kg$

As the angle increase, the safe point on the ladder goes up in a non-linear way. For given conditions, At about 70^0 it is safe to stand at the top of the ladder.



12 The Law of Gravitation

This law has been called "the greatest generalization achieved by the human mind". For centuries scientists and curious minds tried to understand the motions of planets among the stars, and causes behind the motion. Ancient Greeks deduced from simple observations that planets went around the sun. However, they were not able to identify exactly how the planets went around the sun, with exactly what causes. By the beginning of the fifteenth century there were heated debates as to whether the planets really went around the sun or not. These debates were mostly philosophical. It was Tycho Brahe, a Danish astronomer, who suggested to resolve the dispute by accurate measurements of positions of the planets. After a couple decades of careful observations he published extensive tables of data. Years later a mathematician Johannes Kepler derived three laws of planetary motion based on Brahe's tables of data. 1) All planets move in elliptical orbits with the Sun at one focal point, 2) the radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals, and 3) the square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit. However still no explanation was provided for causes of motion. Decades later, in 1686, Isaac Newton published his law of gravitation

12.1 Newton's law of gravitation

The Law of Gravitation is that two bodies attract each other with a force upon each other which varies inversely as the square of the distance between them, and varies directly as the product of their masses. Mathematically we can write that great law as a simple formula

$$F = G\,\frac{m_1 m_2}{r^2}, \tag{12.1}$$

where m_1 and m_2 are masses of bodies, r is the distance between them, and G is a fundamental proportionality constant called the universal gravitational constant or simply gravitational constant. In SI unites its value is

$$G = 6.67408 \times 10^{-11}\, N \cdot m^2/kg^2$$

Please note that this is a recommended value by International Council for Science: Committee on Data for Science and Technology (CODATA) as of 2014. Most standard textbooks, as well as Wikipedia, use $G = 6.67384 \times 10^{-11} N \cdot m^2/kg^2$ that is a value based on less precise measurements. It is also interesting to note that it took 112 years, after publishing the law of gravitation by Newton, to measure G for the first time.

We can also write the law in a vector form (remember that forces are vectors)

$$\vec{F}_{21} = G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{21} \tag{12.2}$$

where $\hat{r}_{21}$ is a unit vector in the direction from 1 toward 2 (note that $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$).

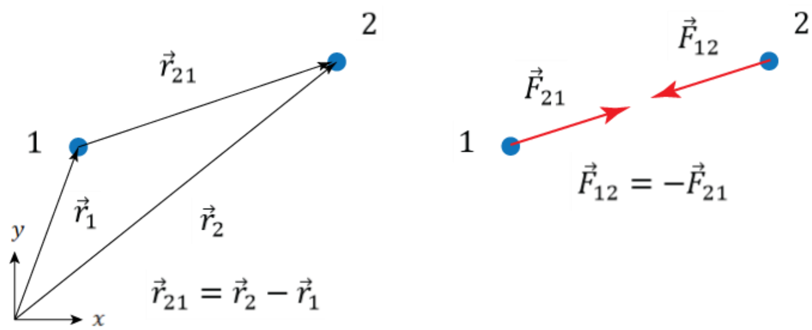


Figure 12.1 Notations for positions and forces

Using the components we can write

$$F_{21,x} = G\frac{m_1 m_2}{r_{21}^3}(x_2 - x_1), \qquad F_{21,y} = G\frac{m_1 m_2}{r_{21}^3}(y_2 - y_1), \qquad F_{21,z} = G\frac{m_1 m_2}{r_{21}^3}(z_2 - z_1).$$

In case of three or more objects we simply use the superposition of forces to find a net force on every object in a system. For example, a net force on object 1 from n other objects is written as

$$\vec{F}_{i1} = \sum_{i=2}^{n} G\frac{m_1 m_i}{r_{i1}^2}\hat{r}_{i1}$$

Having the law of gravitation together with second newton's law we have everything required to derive many consequences of these two principles. The problem of motion of objects interacting by the law of gravity can analytically be solved only for two objects. For three or more objects the problem can be treated only numerically.

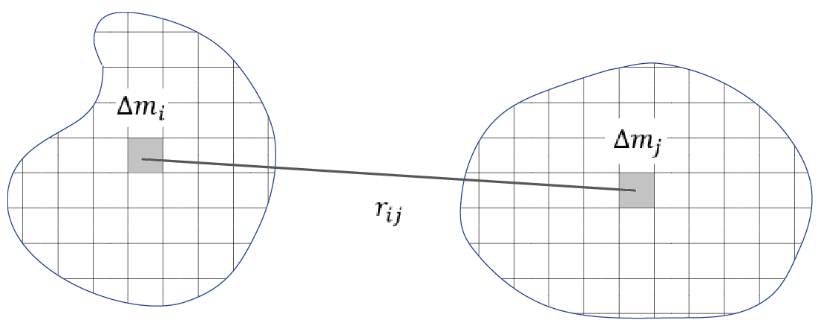
It is amazing that the fundamental law of gravity, looking so simply elegant, explains motion of stars, planet, comets and satellites. Besides, any object on a planet is a subject of the gravitational force following the same law of gravity. There is something even more astonishing. The fundamental law of gravity is valid for every object in the universe.

12.1.1 Law of gravitation and physical objects

The law of gravitation (12.1) is formulated for point-like objects. We know very well from our experience that planets, stars, people, textbooks are not point-like objects. Therefore we need to do additional work to apply the gravitational law to objects of arbitrary shapes or consider conditions when a point-like approximation is a good one.

The point like approximation works very well if a distance between centers of masses of objects is much larger than physical sizes of the objects. In particularly, for the solar system it is a good approximation (the radius of the sun $R_{Sun} = 6.955 \times 10^8 m$ and the distance to the closes planet Mercury $6.532 \times 10^{10} m$, with distance to Earth $1.485 \times 10^{11} m$).

In a general case (when the point-like approximation cannot be applied) we can divide two objects I and J into small elements Δm_i and Δm_j.



Then every small element of object I attracts every small element of object J. For every pair of elements we apply the law of gravity

$$\Delta f_{ij} = G \frac{\Delta m_i \Delta m_I}{r_{ij}^2}$$

The net force of gravitational interaction between two objects is a *vector sum* of all elementary forces

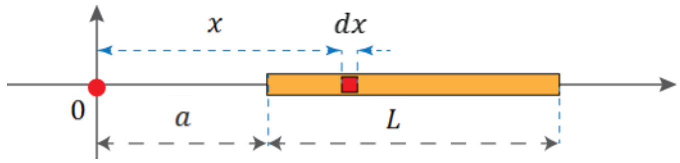
$$\vec{F}_{12} = G \sum_i \sum_j \frac{\Delta m_i \Delta m_j}{r_{ij}^2} \hat{r}_{ij}$$

In the limit when $\Delta m_i \to 0$ and $\Delta m_j \to 0$ the sum above is reduced to multiple integration. In particular, a force between a particle of mass m and an extended object of mass M with a density distribution $\rho(r)$ can be written as

$$\vec{F} = Gm \int \frac{dM}{r^2} \hat{r} = Gm \int_V \frac{\rho(r) dV}{r^2} \hat{r} \qquad (12.3)$$

where the integration is carried out over the volume of an object.

Let us consider a simple application, namely a force of gravity between a homogeneous bar of length L and mass M and a particle of mass m located at distance a from the bar.



Keeping in mind that dM can be written as $dM = (M/L)dx = \lambda dx$, where λ is a linear density, we can write the total gravitational force exerted by the bar on the particle as

$$\vec{F} = Gm\int_a^{a+L}\frac{dM}{x^2}\hat{\imath} = Gm\int_a^{a+L}\frac{\lambda dx}{x^2}\hat{\imath} = Gm\int_a^{a+L}\left(\frac{M}{L}\right)\frac{dx}{x^2}\hat{\imath} =$$

$$\frac{GmM}{L}\hat{\imath}\int_a^{a+L}\frac{dx}{x^2} = \frac{GmM}{L}\hat{\imath}(-1)\left(\frac{1}{a+L}-\frac{1}{a}\right)$$

and finally

$$\vec{F} = \frac{GmM}{L}\frac{L}{a(a+L)}\hat{\imath} = G\frac{mM}{a(a+L)}\hat{\imath}$$

We can see that in the limit $L \to 0$,

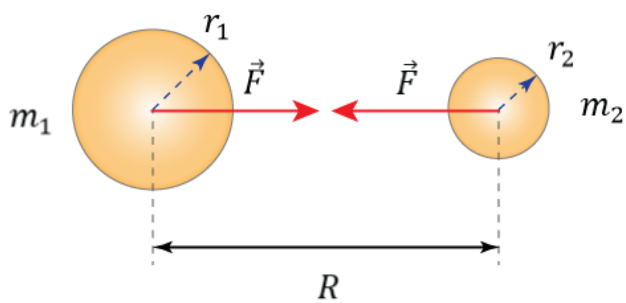
$$\vec{F} = \lim_{L\to 0} G\frac{mM}{a(a+L)}\hat{\imath} = G\frac{mM}{a^2}\hat{\imath}$$

and the force of gravity corresponds to a force between two point-like masses. Additionally, if $a \gg L$ (the distance between the objects is much larger that objects), then again we have it like a force between two particles.

Generally, calculations of multiple integrals, in cases of physical objects of various shapes, can be quite cumbersome. However, for this chapter we only need a couple special cases, namely gravitational force between a star and a planet, between a planet and a satellite, and between a planet and an object on or close to a planet's surface.

12.1.2 Newton's shell theorem

Most planets and starts are spheres (or almost spheres) with *spherically symmetric mass distributions.* Then we can use Newton's shell theorem that states "*A uniform spherical shell of matter attracts a particle that is outside the shell if all the shell's mass were concentrated at its center*". From the shell theorem follows that bodies having a spherically symmetric mass distribution over their volume

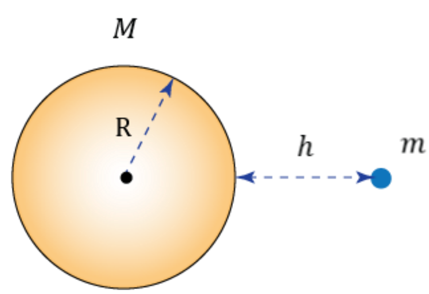


interact as if their masses were concentrated at the centers of these spheres

$$F = G\,\frac{m_1 m_2}{R^2} \tag{12.4}$$

Planets and stars are either far away from each other ($R \gg r_1, r_2$) or have geometric shapes very close to spherical. Therefore equation (12.4) is a very good one for celestial mechanics.

For a physical object outside a uniform solid sphere or spherical shell



the magnitude of gravitational force on this object is

$$F = G\,\frac{mM}{(R+h)^2} \tag{12.5}$$

where R is the radius of a sphere and h is the distance to the sphere's surface from a physics object. If a size of a physics object is much less than $R + h$ then the physics object can be treated as a point like object (its real shape does not matter). Equation (12.5) is a very good approximation for gravitational for between a planet and a satellite and for any objects on or above a surface of a planet.

Near the Earth surface $h \ll R$ then

$$\frac{1}{(R+h)^2} = \frac{1}{R^2}\frac{1}{(1+h/R)^2} \approx \frac{1}{R^2}\left(1 - 2\frac{h}{R} + \cdots\right),$$

where we have neglected the quadratic and higher-order terms in $(h/R)^2$ since h/R is very small. For example, at the top of Everest we have $h/R \approx 1.4 \times 10^{-3}$. In most cases, there is no need to take into account insignificant variations in the force of gravity. Within the accuracy 10^{-3}, we can consider that for objects on Earth the force of gravity is constant and independent of the altitude.

It is instructive to evaluate magnitudes of gravitational force between some objects. Thus the force between Earth and Sun is $3.53 \times 10^{22} N$ (quite a large number!), the force between a human body and Earth is simply its weight, for a 60 kg body it is about 590 N, the force between two humans (assume they have spherical shapes) separates by a distance of 10 meters is $2.4 \times 10^{-9} N$ (or about 10,000 times less than a weight of mosquito). As we can see gravitational force between objects on Earth is very small comparing to their weights.

Some people believe that positions of stars at birth influence somebody's life. Assume that they are taking about gravitational force. Let's calculate gravitational force from the closest star outside the solar system (Sirius) on a human body at birth, we can easily get it is about $1.6 \times 10^{-7} N$, or about ten times less than one eyelash hair. (### check these numbers)

12.1.3 Gravitational and Inertial Masses

Expressing the law of gravity (12.1) we silently assumed that masses in the law are the same masses that we have in the second newton's law (4.5). However this assumption is not solid without additional insight. The law of gravity and the second newton's law are two independent laws. In the second newton's law mass characterizes the property of inertia of motion, or inertial mass is the property of an object that measures the object's resistance to acceleration. In the law of gravity mass characterizes a property of objects to attract each other with gravitational force. Therefore it is reasonable to ask a question if we should rather assign m_I for the inertial mass, and m_G for the gravitational mass and to treat them differently. It is clear that only experiments can answer this questions. From the second Newton's law follows for the free-fall acceleration

$$g = \frac{F_g}{m_I} = G\frac{M_E}{R_E^2}\frac{m_G}{m_I}$$

where M_E is the mass of Earth, and R_E is the radius of Earth. Experiments show that free fall acceleration g is the same for all objects at the same location. Since G, M_E, R_E are constants for the same locations, then the ratio m_G/m_I must be the same for all objects. At this time highly precise measurement show no difference between the two masses with accuracy 10^{-18}. The equivalence of inertial and gravitational masses (the equivalence principle)

$$m_I = m_G$$

is at the foundation of general theory of relativity.

12.2 Acceleration due to gravity g

In previous chapters we use for the magnitude of the gravitational force on objects from Earth as

$$F = mg$$

Assuming that the earth is a uniform sphere of radius R_E and mass M_E, then a small body of mass m at the earth's surface (a distance R_E from its center) experiences gravitational force

$$F = G\frac{mM_E}{R_E^2} = mg$$

We can easily find that the acceleration due to gravity at the earth's surface

$$g = g_E = G\frac{M_E}{R_E^2} \tag{12.6}$$

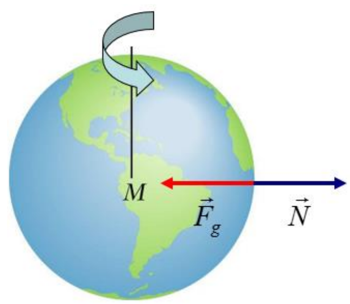
The acceleration due to gravity g is independent of the mass m of the object. By the way, equation (12.6) provided a way to calculate the mass of the Earth since the radius was known since ancient times.

However we need to note that under closer examination the free-fall acceleration g is not a constant. In reality it is a bit different from (12.6) for three reasons.

First, Earth is not uniform. Its density varies radially, and the density of the crust varies slightly from region to region (larger in mountain regions).

Second, Earth is not a perfect sphere but approximately an ellipsoid flattered at the poles and bulging at the equator. Its polar radius is smaller than its equatorial radius by about 21 km (that is 0.33% difference). So, a point at the poles is closer to the dense core of Earth, than a point on the equator, and the free-fall acceleration is large at the poles.

Third, Earth is rotating. Therefore a point on equator has the largest centripetal force due do the rotation. We can easily analyze this effect using our knowledge from previous chapters. Let us consider an object located on the equator.



There are two forces acting on the object, namely the gravitational force and a normal force. We normally assume that the magnitude of the normal force is $N = mg$. From (12.6)

$$g_E = G\frac{M_E}{R_E^2}$$

Then we can write

$$N - mg_E = -\frac{mv^2}{R}, \quad mg - mg_E = -\frac{mv^2}{R}$$

then

$$g = g_E - \frac{v^2}{R} = g_E - \omega^2 R$$

where ω is the angular speed of Earth. Practically the equation above says that free-fall acceleration is gravitational acceleration minus centripetal acceleration. For an object on the equator this centripetal acceleration is very small (about 0.034 m/s^2 compared with 9.8 m/s^2). If the Earth had a perfectly spherical shape, then an object moved from a pole to the equator would “lose” 0.35% of its weight. Accounting for the deviation from the ideal spherical shape of Earth makes such “loss of weight” even larger, namely about 0.5%. Therefore the free fall acceleration changes with latitude from 9.780 m/s^2 at the equator to 9.832 m/s^2 at the poles. For this course neglecting the difference between g and g_E is often well justified. Therefore within the accuracy 10^{-2} we can consider the free-fall acceleration as constant and independent of the altitude and position on Earth. The standard acceleration of gravity recommended by CODATA is

$$g = 9.80665\ m/s^2$$

* This acceleration corresponds to a location at 45 degrees of latitude. For practical purposes in this book we use $g = 9.8\ m/s^2$.

12.3 Gravitational potential energy

In chapter 7 we introduced potential energy for conservative forces, which is which is energy associated with a configuration of objects. The gravitational force (12.1) is a conservative force (no proof here). We already evaluated gravitational potential energy as $U = mg\left(y_f - y_i\right)$. However that evaluation has been done under assumption for the free-fall acceleration to be constant. This assumption is a good one as soon as objects are not too far from Earth’s surface comparing to Earth’s radius. Now we want to derive a more general expression for the gravitational potential energy. A conservative force and potential energy are connected (7.2)

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} F(\vec{r}\,')d\vec{r}\,'$$

We can easily notice that the gravitational force (12.1) on object m is spherically symmetrical

$$\vec{F} = -G\frac{mM}{r^2}\hat{r}$$

where $\vec{r}$ is a vector directed from the origin to the body (that is why we have the negative sign here). Since the integral above is independent of paths we can integrate the force along a radius

$$U(r) = -\int_{r_0}^{r} F(r')dr' = GmM\int_{r_0}^{r}\frac{dr'}{r'^2} = -GmM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

For any potential energy the choice of a reference point is entirely arbitrary. It is common to choose it where the force is zero. Thus setting $U(r) = 0$ at $r = \infty$ where $F(r) \to 0$ we get

$$U(r) = -G\frac{mM}{r} \qquad (12.7)$$

The equation above can be applied to any two particles with masses m and M. The gravitational potential energy between two objects increases with distance because the gravitational force attracts masses.

The gravitational potential energy for a system with more than two particles is the sum over all pairs of particles. For example, for three particles we have

$$U_{total} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$

The absolute value of the gravitational potential characterizes the work needed to separate the particles by an infinite distance.

By inspecting formula (12.7) one may be wondering why do we have two definitions for the gravitational potential energy, namely $U = mgh$ and (12.7). Let us consider a change in the gravitational potential energy of an object raised from the ground of Earth to a height h

$$U(R_E + h) - U(R_E) = -G\frac{mM_E}{R_E + h} + G\frac{mM_E}{R_E} = GmM_E\left(\frac{1}{R_E} - \frac{1}{R_E + h}\right) = GmM_E\frac{h}{R_E(R_E + h)}$$

When the height h is very small compared with the radius of Earth R_E, we can disregard terms h/R, yielding

$$U(R_E + h) - U(R_E) \approx GmM_E\frac{h}{R_E^2} = mh\frac{GM_E}{R_E^2} = mhg$$

since according to (12.6) the last term GM_E/R_E^2 in the equation is free fall acceleration g.

12.4 Motion of planets and satellites

Motion of three or more objects interacting by force of gravity has been a challenging problem for both physicists and mathematicians since Newton. It has been demonstrated that even 3-body problem does not have an analytic solution. There are either infinite series solutions or solutions for specific modes of motion. Interesting enough French mathematician and physicist Poincaré working with the problem laid the foundations of modern chaos theory (the butterfly effect). Thus, motion of celestial (or astronomical) objects can only be calculated numerically[8]. Quite often when we do not need high accuracy we can substitute a many-body problem with a two-body one. If one of the bodies has a mass much larger than the other, then we practically dealing with a one-body problem moving in a gravitational field of a heavy body. If we disregard interaction between the planets in the solar system (a simplified view of the solar system) comparing to their interaction with the Sun, then we approximate their motion as one-body motion around the sun. Using astronomical data you can easily evaluate quality of this approximation by comparing gravitational forces in the solar system.

For the Earth-Sun system we have that the mass of the Sun ($2 \times 10^{30} kg$) is 332,000 times the mass of the Earth ($6 \times 10^{24} kg$). Hence the Sun can be considered to be stationary to a high degree of accuracy, and the Earth can be expected to be revolving around a fixed center. Besides, the Sun and

[8] Millennium Simulation or Millennium Run is likely the most impressive N-body simulation. The Millennium Run was populated by about 20 million "galaxies" to trace the evolution of the matter distribution in a cubic region of the Universe over 13 billion light-years.

the Earth have almost spherical shapes, and also the distance between the Sun and the Earth is much larger than diameters of the Sun and the Earth. Therefore, we can treat them as point-like objects. The other example is motion of satellites orbiting the Earth. Sure, their gravitational interactions is many orders of magnitude less than the force of gravity from the planet.

Equation of motion for an object of mass m (a planet or a satellite) in a field of a heavy object of mass M (a star or a planet accordingly) can be written in the form

$$m\frac{d\vec{v}}{dt} = -G\frac{mM}{r^2}\hat{r} \tag{12.8}$$

where $\vec{r}$ is the radius vector of the planet relative to the mass M. This is a second-order ordinary differential equation. Solving it together with initial conditions provide a position of the planet as a function of time. All Kepler's laws can be derived from such solutions. Such an exercise is a good one for courses of classical mechanics and mathematical physics.

Here are the most important result from analysis of (12.8). The force of gravity acting on a point mass is directed along the radius vector. The moment of this force about the center of force is zero, and for angular momentum L (10.9) we have

$$\frac{dL}{dt} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d}{dt}m\vec{v} = 0$$

Therefore the angular momentum of a point mass in our case has a constant magnitude as well as direction

$$L = \vec{r} \times m\vec{v} = const.$$

For an elementary displacement this equation can be rewritten as

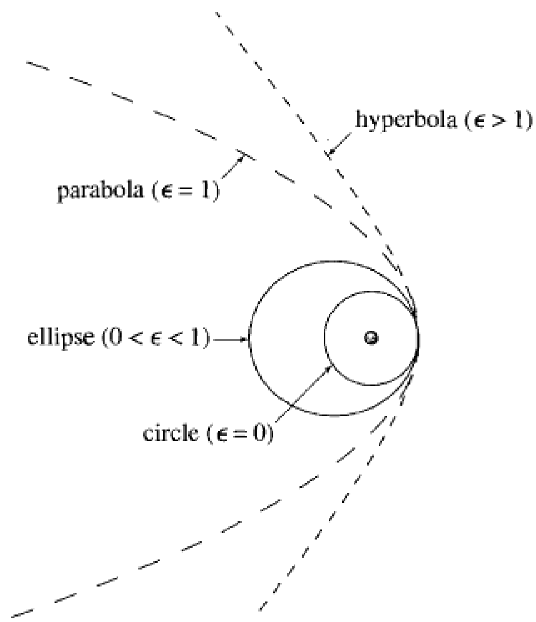
$$L = m\vec{r} \times \vec{v} = m\vec{r} \times \frac{d\vec{r}}{dt}$$

Hence the elementary displacement $d\vec{r}$ and the radius vector $\vec{r}$ are in a plane perpendicular to the angular momentum $\vec{L}$. This means that all the time the motion happens in the same plane.

In polar coordinates (r, φ) the general solution (Kepler orbit) is given by

$$r(\varphi) = \frac{c}{1 + \epsilon \cos\varphi}$$

where c is a constant and ϵ is the eccentricity. There are four different Kepler orbits (depending on initial conditions). For total energy $E < 0$ the solutions are bounded orbits either a circle ($\epsilon = 0$) or an ellipse ($0 < \epsilon < 1$). For total energy $E \geq 0$ the solution are either a parabola ($E = 0, \epsilon = 0$) or a hyperbola ($E > 0, \epsilon > 1$).



12.5 Planets and satellites: circular orbits, escape speed.

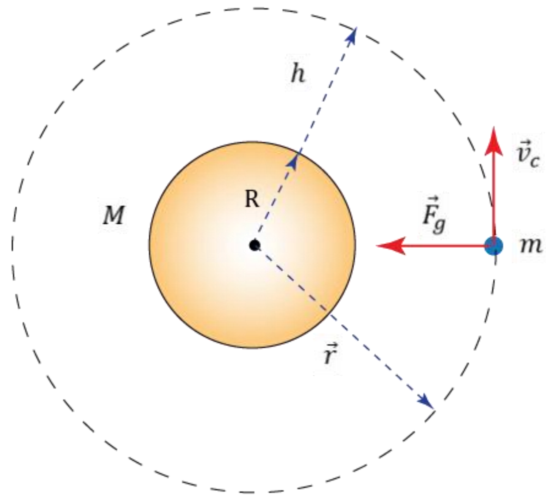
Since solving ordinary differential equation goes well beyond this course, we are going to consider a very simple, but quite practical case or motion, namely a uniform circular motion under gravity. In this case it is gravitational force that works as a centripetal force for uniform circular motion. Consider a planet of mass m moving around the Sun of mass M in a circular orbit with r a distance between their centers of masses and $M \gg m$,

$$G\frac{mM}{r^2} = \frac{mv_c^2}{r}$$

or the orbital speed v_c of the planet is related to its circular orbit as

$$v_c = \left(G\frac{M}{r}\right)^{1/2} \tag{12.9}$$

Note that the orbital speed does not depend on the plant's mass, and is simply a function of its orbital radius. That means that any object inside or outside a satellite moves with the same velocity as the satellite without even touch the satellite. For example, astronauts on board of the International Space Station (ISS) move with the same velocity as the station, so nothing pushes them against the walls of the station. This state is called a state of *apparent weightlessness*. This is the same as being in a freely falling elevator. Note that the free fall acceleration at the height of the ISS is 0.885*g (pretty far from being true weightlessness).



Since the period of revolution is circumference divided by speed, then

$$T_c = \frac{2\pi r}{v}$$

Substituting the orbital speed from (12.9) the preceding expression becomes

$$T_c = \frac{2\pi r^{3/2}}{(GM)^{1/2}} \tag{12.10}$$

We can also apply equation (12.10) to satellite motion around a planet if we consider M as mass of a planet, and m as mass of a satellite.

Quite often for satellite motion it is convenient to denote the distance between the centers of masses r as $r = R + h$ where R is the radius of the planet, and h is a distance form surface of the planet to a satellite, then

$$v_c = \left(G\frac{M}{R+h}\right)^{1/2} \qquad T_c = \frac{2\pi(R+h)^{3/2}}{(GM)^{1/2}} \tag{12.11}$$

Using (12.11) we can calculate the orbital speed and the period for the International Space Station with $h = 250$ miles or about 400 km that gives us $v_c = 7670\ m/s$ and $T_c = 92$ minutes. For the Earth orbiting the Sun we get for the orbital velocity $v_c = 30\ km/s$ (or about 70,000 mph), the period (a year) is 365.3 days.

Some communication satellites are moving in a circle in the earth's equatorial plane. They are at such height that they always remain above the same point. Such orbits are called as geosynchronous orbits. Using 24 hours for the period of revolution we can find from (12.11)

$$h + R = \left(\frac{T_c^2 GM}{4\pi^2}\right)^{1/3}$$

that such satellites must be placed at about h=35,800 km above the Earth surface.

Since we assumed that $M \gg m$ (the heavier object is located at the origin and does not move) then then the total mechanical energy E of the two-body system is

$$E = \frac{1}{2}mv^2 - G\frac{mM}{r}$$

Using (12.9) for a circular orbit gives

$$E = \frac{1}{2}G\frac{mM}{r} - G\frac{mM}{r} = -G\frac{mM}{2r}$$

One can see that for circular orbits the kinetic energy is equal to one-half the absolute value of the potential energy. The absolute value of total energy E is binding energy of the system. It is this amount of energy is needed for the system to separate the two masses infinitely. Thus we can derive another characteristic speed, also called escape speed. This is the minimum value of the initial speed needed to let the object of mass m to escape the gravitational force of object with mass M (for example for a satellite to leave the Earth without coming back). Let at the surface of the Earth the initial speed is v_i and $r = R_E$. When the satellite approaches $r \to \infty$ its speed $v_f \to 0$, thus $K_f \to 0$ anf $U_f \to 0$ Using conservation of energy gives

$$\frac{1}{2}mv_i^2 - G\frac{mM_E}{R_E} = 0$$

and we obtain

$$v_i = v_{esc} = \left(2G\frac{M}{R_E}\right)^{1/2} \tag{12.12}$$

Note that the escape speed is independent of the mass of the object, and independent of the direction of the velocity (but sure ignores air resistance). Equation (12.12) can be applied to any object launched from any planet.

*** add following topics

- How to change an orbit of a satellite
- Black holes and escape speed
- Tides: Moon-Earth interaction
- Shell theorem – gravity inside a planet
- Calculating gravitational force from a physical object – much easier to calculate the potential, then force
- Dark matter

12.6 Examples

Example 12-1

A satellite with a mass of 1000 kg is placed in Earth orbit at an orbit of 500 km above the surface. Assuming a circular orbit,

a) How long does the satellite take to complete one orbit?
b) What is the satellite's orbital speed?
c) What is the minimum energy necessary to place this satellite in orbit (assume there is no air resistance)?
d) For how many miles would this energy power an automobile? (Assume 1 gal. of gasoline produces $1.1*10^8$ J of energy, and this energy is sufficient to operate a car for 25 miles)

Example 12-2

Two spherical objects of masses $m_1$and m_2 are released from rest at a separation distance of L. Find their speeds and positions when their separation distance is r. Assume that the first particle initially is at the origin.

Example 12-3

The fastest possible rate of rotation of a planet is that for which the gravitational force on material at the equator just barely provides the centripetal force needed for the rotation. Calculate this rotation period for Earth.

Example 12-4

The Schwarzschild radius is a distance from a black hole where the escape velocity equals the speed of light (even light cannot escape from a black hole). Determine the Schwarzschild radius for a black hole with the mass of the Sun.

Example 12-5

Consider an asteroid of a size of Texas (see Armageddon movie). Assume that the asteroid has a spherical shape, its radius is 400 km and density is 5.5 g/cm^3. Estimate speed needed to provide to a baseball to make it a satellite orbiting the asteroid just above the surface.

Example 12-6

Calculate the time taken by the Earth to fall from the orbit onto the Sun if its instantaneous radial velocity became zero.

13 Periodic Motion

We encounter periodic motion in both our everyday lives and in science and engineering. Periodic motion is also a basis for any time keeping device, from the Earth rotation (24 hours) to atomic clocks. We also use vibration, oscillation, or harmonic motion as synonyms for periodic motion.

In this chapter we are going to study periodic motion caused by a force that we know explicitly. Thus we can understand all the details from first principles.

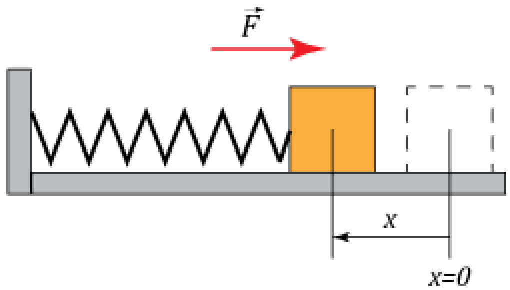
13.1 Simple harmonic motion

There are remarkable equations in physics which appear in various branches of physics, so that vary many effects can be described by same equations. Very many of such equations are linear differential equations with constant coefficients. Examples of phenomena described by such equations are the oscillations of a mass on a spring; the oscillations of current in an electrical circuit.

The simplest mechanical system whose motion follows a linear differential equation with constant coefficients is a mass on a spring. Such a system is also called a harmonic oscillator. Consider a block of mass m attached to the end of a horizontal spring. The block is free to move on a frictionless surface. In this case we have only one force affecting the motion, namely the spring force.

$$\vec{F} = -k\vec{x}$$

The minus sign tell us that the spring force is a restoring force pulling always back in the direction of the equilibrium and opposite to the displacement.



Thus the second Newton's law, mass times the acceleration, must equal to $-kx$

$$m\frac{d^2x}{dt^2} = -kx \tag{13.1}$$

There is a more common form for this equation, namely

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{13.2}$$

where, in case of the spring force

$$\omega = \left(\frac{k}{m}\right)^{1/2} \tag{13.3}$$

Equation (13.2) is called a simple harmonic motion equation (SHM), and it plays an extraordinary role in practically all fields of physics. There are two major reasons for this equation to be the exceptional one. First, harmonic motion is caused by a force that is linear proportional to the displacement. For small displacements any force can be expanded into a Taylor series, where the first linear term, is a leading one. Therefore, physics of a simple harmonic motion is often the first step in studying many periodic motions. Second, many oscillating systems "resonate" to external harmonic oscillations when their frequencies of oscillations are close.

Mathematically, the SHM differential equation has a very simple general solution as

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t \tag{13.4}$$

or in the equivalent form

$$x(t) = A\cos(\omega t + \varphi) \tag{13.5}$$

it is easy to test these solutions by a simple substitution of (13.5) into (13.2).

Let us first analyze the meanings of all terms in the solution (13.5). The constant ω is called the *angular frequency* of the motion (units - radians per second). It is the number of radians by which the phase changes in a second. That is determined by the differential equation. We know that the trigonometric function $\cos x$ is periodic and repeats itself every time ωt increases by 2π. Then

$$\omega t + 2\pi = \omega(t + T)$$

where T is the period of the motion, and hence

$$T = \frac{2\pi}{\omega} \tag{13.6}$$

And for a spring it becomes

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{13.7}$$

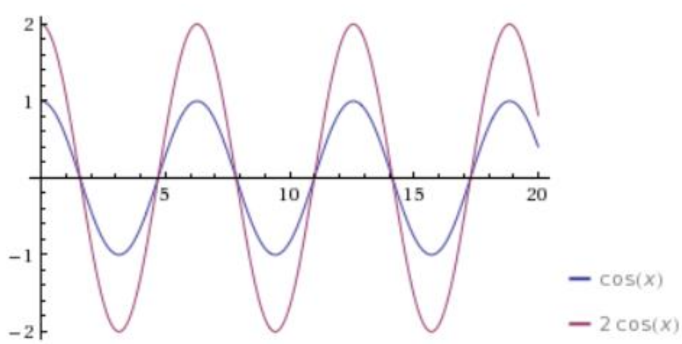
It is very common to also use frequency f of the motion. The frequency represents the number of oscillations that the particle makes per unit time

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{13.8}$$

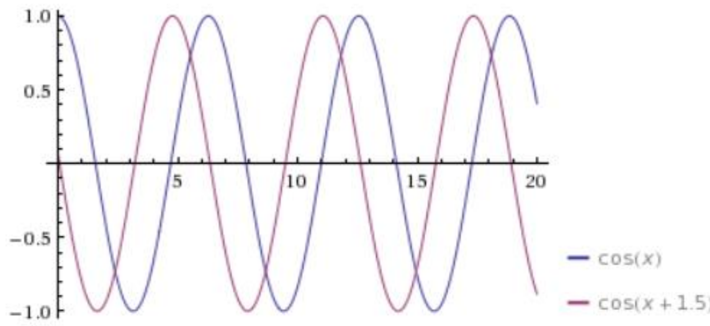
The units for the frequency are cycles per second (s^{-1}) or hertz (Hz). For the angular frequency:

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{13.9}$$

The other constants of motion (A and φ) are not determined by the equation, but by how the motion is started. For a second order differential equations we need two initial conditions to define a specific solution. Very often it is done by setting an initial position and velocity at some initial time, i.e. $x(t = 0) = x_0,\ v(t = 0) = v_0$. Of these constants, A measures the maximum displacement attained by the mass, and is called the *amplitude of oscillation*. We can see that by plotting a graph for two values of A namely A and $2A$.



The constant φ is sometimes called the phase of the oscillation. But that is a confusion, because other people call $\omega t + \varphi$ as the phase, and say the phase changes with time. We might say that φ is a *phase shift from some defined zero*. The following graph should help in visualizing this



We can obtain the linear velocity of a particle undergoing simple harmonic motion by differentiating (13.5) with respect to time

$$v(t) = \frac{dx}{dt} = -A\omega \sin(\omega t + \varphi) \tag{13.10}$$

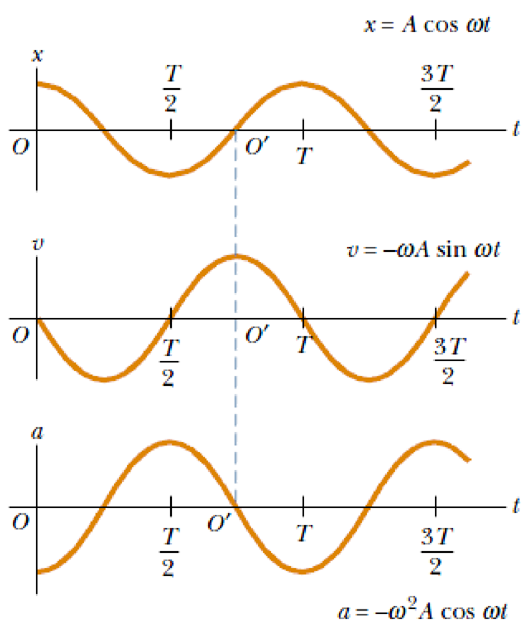
And the acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \varphi) \tag{13.11}$$

Because $x(t) = A\cos(\omega t + \varphi)$ we can write (13.11) as

$$a(t) = -\omega^2 x(t)$$

it is instructive to plot the displacement, velocity and acceleration to compare their evolutions with time.



We can see that the velocity has the largest values $\pm\omega A$ when the displacement$vx = 0$. The acceleration reaches the largest values of $\pm\omega^2 A$ when the displacement has its largest values $x = \pm A$. It is understandable since the restoring force is the largest at those displacements.

13.1.1 Initial conditions

Now let us consider what determines the constants A and φ or C_1 and C_2. As we said before, these constants are determined by how we start the motion, not by any other features of the situation. These are called the initial conditions. We would like to connect the initial conditions with the constants. It is a bit easier to do it for the form (13.4) but we will do it for more often used form (13.5). Suppose that at $t = 0$ we have started with an initial displacement x_0 and a certain velocity v_0. The displacement and velocity in this case

$$\begin{aligned} x_0 &= A\cos\varphi \\ v_0 &= -A\omega\sin\varphi \end{aligned}$$

Hence the initial phase is

$$\varphi = \operatorname{atan}\left(-\frac{v_0}{\omega x_0}\right)$$

From the initial conditions

$$\begin{aligned} \cos\varphi &= \frac{x_0}{A} \\ \sin\varphi &= -\frac{v_0}{A\omega} \end{aligned}$$

Since $\cos\varphi^2 + \sin^2\varphi = 1$

$$\frac{x_0^2}{A^2} + \frac{v_0^2}{A^2\omega^2} = 1$$

and finally

$$A = \left(x_0^2 + \frac{v_0^2}{\omega^2}\right)^{1/2}$$

In a special case when the mass starts with zero initial velocity $v(t = 0) = 0$ we get

$$A = x_0, \quad \varphi = 0$$

Then we see that for this setup, the amplitude is equal to the maximum deviation from the equilibrium.

13.2 Energy of the simple harmonic motion

Now we want to check the conservation of energy. Since there are no frictional losses, energy ought to be conserved. Let us write again the formulae

$$\begin{aligned} x(t) &= A\cos(\omega t + \varphi) \\ v(t) &= -A\omega\sin(\omega t + \varphi) \end{aligned}$$

For a spring, the potential energy at any moment is $kx^2/2$, where x is the displacement and k is the constant of the spring. The kinetic energy is $mv^2/2$. If we substitute for x and v for our expressions above, we get

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t + \varphi) + \frac{1}{2}kA^2 \cos^2(\omega t + \varphi)$$

Since $\omega^2 = k/m$ - see (13.3)

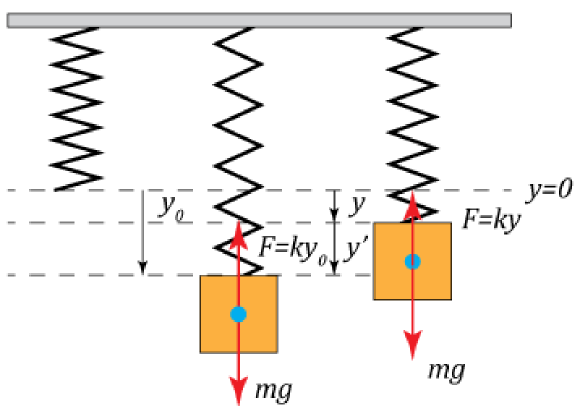
$$E = K + U = \frac{1}{2}m\omega^2 A = \frac{1}{2}kA^2 \tag{13.12}$$

The energy is dependent on the square of the amplitude; if we have twice the amplitude, we get an oscillation which has four times the energy. From the properties of the trigonometric functions follows that the maximum potential energy stored in the spring when there is no kinetic energy. And at the equilibrium position the total energy of the system is in the form of kinetic energy.

13.3 Applications of simple harmonic motion

13.3.1 Vertical spring

So far we considered a horizontal motion of a body attached to a spring, practically having only one force involved in the motion. Naturally it is reasonable to see if we have a simple harmonic motion for a vertical spring, when both the spring force and gravity are involved.



let us choose the downward direction as the positive direction, then the spring's force in the equilibrium position with the mass m attached is

$$F_{net,y} = -ky_0 + mg = 0$$

When the body oscillates and at the position y then

$$F_{net,y} = -ky + mg = -k(y_0 - y') + mg = -ky_0 + ky' + mg = ky'$$

Note that we used $ky_0 = mg$. The equation of the motion (where

$$m\frac{d^2y}{dt^2} = m\frac{d^2(y_0 - y')}{dt^2} = -m\frac{d^2y'}{dt^2} = ky'$$

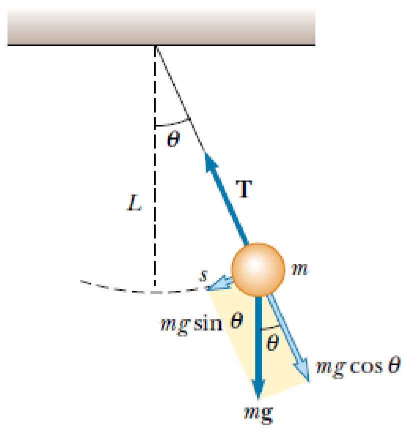
or

$$\frac{d^2y'}{dt^2} = -ky'$$

which is the same as the equation for the horizontal spring (13.1) with y'substituting x. The only physical change is that the equilibrium position y_0 corresponds to the point at which $ky_0 = mg$. The effect of the gravitational force is simply shifting the equilibrium position from $y = 0$ to $y_0 = mg/k$. The angular frequency of oscillation and the period are the same as for a sprong with a horizontal orientation.

13.3.2 The simple pendulum

The simple pendulum consists of a particle-like mass m (also called a bob) suspended by a massless, unstretchable string of length L. The motion occurs in the vertical plane and is driven by the force of gravity.



There are two forces acting on the mass, namely the tension in the string T and the gravitational force mg. The tangential component of the gravitational force $mg\sin\theta$ at all times acts opposite the displacement. In this case Newton's second law for rotational motion (10.8)

$$I\frac{d^2\theta}{dt^2} = \tau$$

Since for a point-like mass on a string $I = mL^2, \tau = Lmg\sin\theta$ we have

$$mL^2\frac{d^2\theta}{dt^2} = -Lmg\sin\theta$$

This equation can be rewritten as

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

that does not look like a SHM equation. However, for small angles when $\theta \ll 1$ (in radians) we Taylor series as

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + O(\theta^7)$$

Keeping only the first linear term gives equation of motion for the simple pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{13.13}$$

With

$$\omega = \left(\frac{g}{L}\right)^{1/2}$$

this is exactly equation for simple harmonic motion with the solution

$$\theta = \theta_m \cos(\omega t + \varphi) \tag{13.14}$$

The period of the motion is

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{13.15}$$

Thus, the period of a simple pendulum depend only on the length of the string and the acceleration due to gravity.

It is reasonably to ask what small angle (in degrees) is a good small angle approximation. Simple calculations show that for 14^0 angle we get 1% accuracy for the linear approximation, and we need a higher accuracy then 6^0 angle gives 0.2% accuracy.

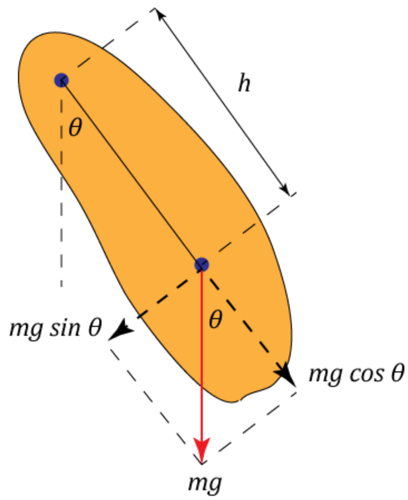
Note that from (13.15) follows

$$g = (2\pi)^2\frac{L}{T^2} \tag{13.16}$$

and this result can be used for very precise measurement of free fall acceleration.

13.3.3 The physical pendulum

A physical (real) pendulum can be a complicated distribution of mass. Consider a rigid body pivoted at a point that is a distance h from the center of mass. The force of gravity provides a torque about an axis through. The magnitude of the torque is $mg\sin\theta$.



Second newton's law for rotation reads

$$I\frac{d^2\theta}{dt^2} = \tau = -mgh\sin\theta$$

Using, again, the small angle approximation we can rewrite the above equation

$$\frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\theta$$

We can see that for small angular displacements the physical pendulum is follows simple harmonic motion with

$$\omega = \sqrt{\frac{mgh}{I}}$$

and the period of oscillations

$$T = 2\pi\sqrt{\frac{I}{mgh}} \tag{13.17}$$

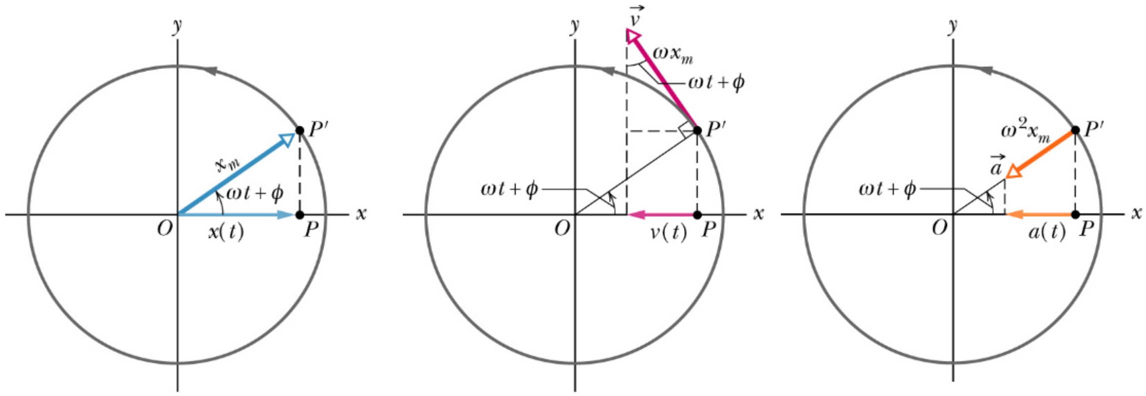
Equation (13.17) reduces to the period of a simple pendulum (13.15) when $I = mL^2$, or when all the mass is concentrated at the center of mass. This result can be used to measure the moment of inertia of a flat rigid body if the location of the center of mass is known. The moment of inertia can be obtained by measuring the period.

13.4 Simple harmonic motion and circular motion

The fact that the solution $x(t) = A\cos(\omega t + \varphi)$ contains cosines assumes that there might be connection to a circular motion. Of course, such connection, if exists, is artificial because there are two different kinds of motion. However, let us consider a uniform circular motion of a particle moving in a circle of radius x_m with a constant speed v. The angular position of a particle is θ that changes with time as $\omega t + \phi$. Then $d\theta/dt = \omega = v/x_m$. We know that there is an acceleration $a = v^2/x_m = \omega^2 x_m$ toward the center. At a given moment t the positions along the x and y are

$$x = x_m \cos\theta\,, \quad y = x_m \sin\theta$$

Using geometry and trigonometry we can write for the projection on the x axis



$$x(t) = x_m \cos(\omega t + \phi)$$
$$v(t) = -\omega x_m \sin(\omega t + \phi)$$
$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

which is exactly equations (13.5), (13.10) and (13.11).

Because uniform circular motion is so closely related mathematically to oscillatory up-and-down motion, we can analyze oscillatory motion in a simpler way if we imagine it to be a projection of something going in a circle.

13.5 Damped and forced oscillations*

So far we considered oscillations free from frictional forces. However, a spring or a pendulum eventually stops because the mechanical energy is dissipated by friction (fluid/air resistance). Such motion is said to be damped. For relatively slow speeds the frictional force of fluid resistance can be expressed as

$$F_d = -av$$

where a is a damping coefficient. The minus sign indicates that the force opposes the motion.

13.5.1 Damped motion

We consider a body of mass m attached to a spring and oscillating horizontally about its position of equilibrium. Let x be the horizontal distance of the body from the equilibrium position. Suppose that the motion takes place in a medium whose resistance is proportional to the velocity. The differential equation of motion becomes:

$$m\frac{d^2x}{dt^2} = -a\frac{dx}{dt} - kx = 0$$

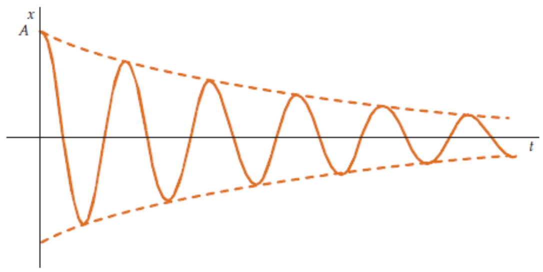
or with notations $2b = a/m$ and $\omega_0^2 = k/m$ we can write

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

There are two types of solutions depending on the value of $D = b^2 - \omega_0^2 < 0$. Most interesting motion happens when the coefficient of resistance b is fairly small compared with the coefficient of restoration k, so that $b^2 - \omega_0^2$ is negative: $b^2 - \omega_0^2 = -\omega'^2$.

Then the solution can be written in the form

$$x(t) = Ae^{-bt}\cos(\omega' t + \varphi)$$



where

$$\omega' = \sqrt{\omega_0^2 - b^2}$$

Here, ω' is the angular frequency of free vibrations, A is the initial amplitude, and φ is the initial phase. This formula represents damped oscillations, the speed of damping being characterized by the factor e^{-bt}. In an interval of time equal to the period, the amplitude decreases in the ratio $e^{-b\tau}$.

When $\omega' = 0$, the condition is called critical damping. The system no longer oscillates but returns to its equilibrium position without oscillation as

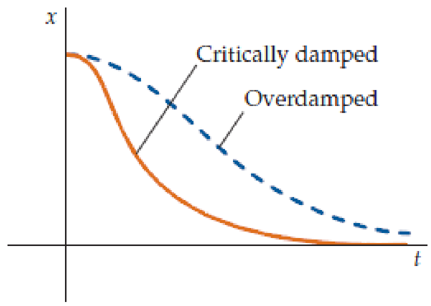
$$x(t) = e^{-bt}(C_1 + xC_2)$$

where C_1 and C_2 are defined by initial conditions.

If $b^2 - \omega_0^2 = \gamma^2$ is positive, the solution is

$$x(t) = C_1 e^{(-b+\gamma)t} + C_2 e^{-(b+\gamma)t}$$

Since we obviously have here $\gamma < b$, therefore x tends to zero on indefinite increase of t but the system returns to equilibrium more slowly than with critical damping.



13.5.2 Forced oscillations and resonance

Assume that there is an additional external force $f(t) = f_0 \sin(\omega t)$ acting on the block. Then we can modify the dumping equation as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \sin(\omega t)$$

This differential equation can be solved either by variation of parameters or using the method of undetermined coefficients. This subject belong to courses on ordinary differential equations. The general solution can be written as

$$x = C_1 e^{-bt} \cos\beta t + C_2 e^{-bt} \sin\beta t + \frac{f_0}{(\omega_0^2 - \omega^2)^2 + (2b\omega)^2}\{(\omega_0^2 - \omega^2)\sin\omega t - 2b\omega\cos\omega t\}$$

The solutions shows that, because an external force is driving it, the motion is not damped. The external force provides the necessary energy to overcome the losses due to the retarding force. Note that the system oscillates at the angular frequency ω of the driving force. For small damping, the amplitude becomes very large when the frequency of the driving force is near the natural frequency of oscillation. The dramatic increase in amplitude near the natural frequency ω_0 is called resonance.

13.6 Examples

Example 13-1

In what time after motion begins will a harmonically oscillating point be brought out of the equilibrium position by half the amplitude? The oscillation period is 24 s and the initial phase is zero. Answer: 4 s

Example 13-2

The initial phase of harmonic oscillation is zero. What fraction of the period will it take for the velocity of the point be equal to half its maximum velocity? Answer t=(1/6)T

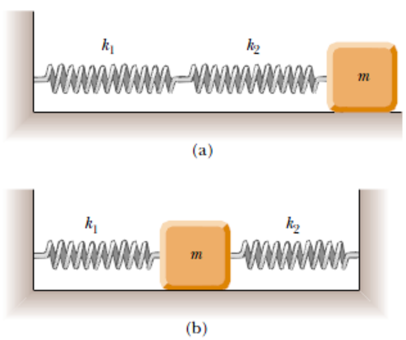
Example 13-3

A ball suspended from a thread 2 m long is deflected through an angle of 4 degrees and its oscillations are observed. Assuming the oscillations to be undamped and harmonic, find the velocity of the ball when it passes through the position of equilibrium. Check the solution by finding this velocity from the equations of mechanics. Answer: 0.31 m/s

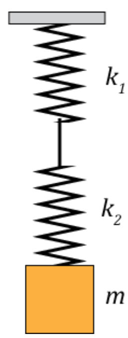
$$v = \theta_0\sqrt{gl}$$

Example 13-4

A mass m is connected to two springs of force constants k_1 and k_2 , In each case, the mass moves on a frictionless table and is displaced from equilibrium and then released. Find the periods of motion for both configurations



As small variation of the above problem: Find the period of small vertical oscillations of a body with mass m for the system



Example 13-5

How will the period of vertical oscillations of a load hanging on two identical springs change if instead of tandem connection the springs are connected in parallel? Answer: The period will be halved.

Example 13-6

A particle of mass m is located in a unidimensional potential field where the potential energy of the particle depends on the coordinate x as $U(x) = U_0(1 - \cos ax)$; U_0 and a are constants. Find the period of small oscillations that the particle performs about the equilibrium position.

Answer: $2\pi\sqrt{m/a^2U_0}$

Solve the foregoing problem if the potential energy has the form

$$U(x) = \frac{a}{x^2} - \frac{b}{x}$$

where a and b are positive constants. Answer $4\pi a\sqrt{ma/b^2}$

Example 13-7

A body of mass m fell from a height h onto the pan of a spring scale. The masses of the pan and the spring are negligible, the spring coefficient is k. Having stuck to the pan, the body starts performing harmonic oscillations in the vertical direction. Find the amplitude and the energy of these oscillations.

$$A = \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}}, \qquad E = mgh + \frac{1}{2}\frac{m^2g^2}{k}$$

(b) Solve the problem for the case of the pan having a mass M. Find the oscillation amplitude in this case.

Example 13-8

A physical pendulum is positioned so that its center of gravity is above the suspension point. From that position the pendulum started moving toward the stable equilibrium and passed it with an angular velocity ω. Neglecting the friction find the period of small oscillations of the pendulum.

Answer: $T = 4\pi/\omega$

Example 13-9*

A pendulum clock is mounted in an elevator car which starts going up with a constant acceleration a, with a < g. At a height h the acceleration of the car reverses, its magnitude remaining constant and the elevator car goes another h to stop. Compare time measured by the pendulum clock to time measured by not moving clock

$$T_{pend} = T_0\left(\sqrt{\frac{g}{g-a}} + \sqrt{\frac{g}{g+a}}\right)$$

Example*

Imagine a shaft going all the way through the Earth from pole to pole along its rotation axis. Assuming the Earth to be a homogeneous ball and neglecting the air drag, find: (a) the equation of motion of a body falling down into the shaft; (b) how long does it take the body to reach the other end of the shaft; (c) the velocity of the body at the Earth's center.

Answers:

$$\frac{d^2y}{dt^2} = -\frac{g}{R}y, \quad T = \pi\sqrt{\frac{R}{g}} \;\; 42\,min, \;\; v = \sqrt{gR} = 7.9\,km/s,$$

<u>Example*</u>

A mathematical pendulum 0.5 m long brought out of equilibrium deflects by 5 cm during the first oscillation and by 4 cm during the second one (in the same direction). Find the time of relaxation, i.e., the time during which the amplitude of the oscillations decreases e times, where e is the base of natural logarithms.

6.4 s.

<u>Example*</u>

A pendulum clock is mounted in an elevator car which starts going up with a constant acceleration a, with a < g. At a height h the acceleration of the car reverses, its magnitude remaining constant. How soon after the start of the motion will the clock show the right time again?

$$t = \sqrt{\frac{2h}{a}}\frac{\sqrt{1+\mu} - \sqrt{1-\mu}}{1-\sqrt{1-\mu}}, \;\; \mu = a/g$$

A pendulum clock is placed inside a rocket taking off vertically with acceleration a. What will be the reading of the clock after the rocket fall back to the ground, if the engine worked for T seconds?

$T_1 = T\sqrt{1 + a/g}$

???

A particle moves along the x axis as $x = 2.7\cos \pi t$. Find the distance that the particle covers during the time interval from t = 0 to 60 s. (both calculus and thinking)

14 Fluids

It is impossible to imagine life without water and air. They are both fluids. The main property that differentiates fluids from solids is that fluids do not maintain their own shape but conforms to boundaries of a container in which we put them. In other words – a fluid is a substance that flows because it will move under the shear. On molecular level it means that molecules in fluids are held together by weak forces that are not strong enough to maintain a shape under gravity or other external force.

14.1 Density and pressure

So far we mostly work with mass and force. Since fluids are extended substances we need to introduce two properties of fluids. Such properties that they can vary from point to point inside that fluid.

First we need to define an equivalent of a force acting locally on a point in a fluid, and it is called pressure. We express the average pressure p as the *normal* force per unit area

$$p = \frac{\Delta F_{\perp}}{\Delta A}$$

In the limit $\Delta A \to 0$

$$p = \frac{dF_{\perp}}{dA} \tag{14.1}$$

Note that pressure is a scalar quantity. It does not have a specific direction inside a fluid because it acts in all directions. At a surface the pressure acts perpendicular to the surface.

The SI unit of pressure is the *pascal*, where 1 pascal = 1Pa = 1 N/m^2.

Most common units for pressure in everyday life are millibars (mb) or atmosphere (atm) for atmospheric pressure, pound per square inch (psi) in engineering (tire pressure).

Another local characteristic that we need is the density that we define as

$$\rho = \frac{\Delta m}{\Delta V}$$

or in the limit $\Delta V \to 0$

$$\rho = \frac{dm}{dV}.$$

For a fluid with uniform density (like water) we use

$$\rho = \frac{m}{V} \tag{14.2}$$

where m and V are the whole mass and volume of the fluid.

Density is also a scalar quantity. The SI unit of density is the kilogram per cubic meter (1 kg/m^3). Note that the density of a gas varies with pressure, (compressible fluids) but the density of a liquid (e.g. water) does not (uncompressible fluid).

The density of air is about 1.21 kg/m^3. It is interesting to estimate how many kg or lb of air in an average classrooms. We may feel it is a very little number. Assume that a classroom has following dimensions: 12 m deep, 8 m wide and 4 m tall. Than according to (14.2) $m = \rho V = 465\ kg$ (or 1024 lb) which is quite a large number!.

14.2 Hydrostatics

Hydrostatics is the theory of liquids at rest. When liquids are at rest, there are no shear forces. There are two conclusions from this. First, the pressure stress is the same in all directions. Second, the stresses are always normal to any surface inside the fluid. (Here we state both conclusions without proof. But the proof is quite straightforward).

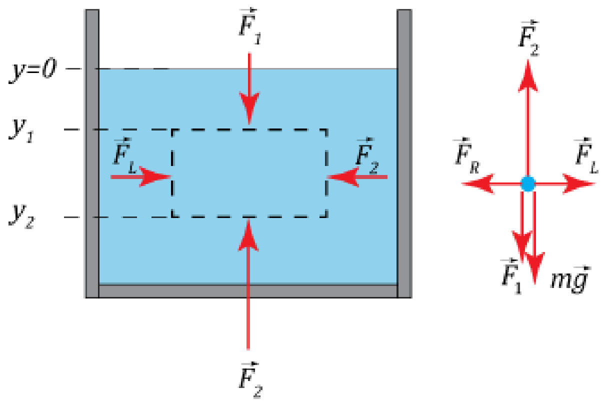
The pressure in a fluid may vary from place to place. From our experience we know that as we dive, even in a swimming pool, we feel water pressure increases with depth.

14.2.1 Hydrostatic pressure

Let us analyze an effect of gravity on pressure in a fluid if the density ρ of the fluid is considered constant (it means liquid – uncompressible fluid).

Let us select a volume of the liquid contained within an imaginary boundaries. Let the cross-sectional area of the top and bottom surfaces is A, and y_1 and y_2 are depths below the surface.

In case of a static equilibrium (hydrostatic) the sampled volume does not move, that if the net forces in all directions are zero. We will concentrate on forces active in the vertical direction since there is no gravity acting in the horizontal directions. Assume that the liquid pressure is p_1 at y_1 level and p_2 at y_2 level.



Then the force acting on the top surface of the volume is $F_1 = p_1 A$ and on the bottom surface if $F_2 = p_2 A$. The net force in the vertical direction is

$$F_{net} = F_2 - F_1 - mg$$

The net force acting in the horizontal direction is zero (fluid is in equilibrium)

$$F_R + F_L = 0$$

The mass of the liquid in the volume is $m = \rho V = \rho A(y_1 - y_2)$. (Note that both y_1 and y_2 are negative). Then together with forces defined through the pressures

$$p_2 A = p_1 A + \rho A g(y_1 - y_2)$$

or

$$p_2 = p_1 + \rho g(y_1 - y_2) \tag{14.3}$$

If we seek for the pressure as a function of depth h (where h is positive) below the liquid surface, then we choose p_0 to be the atmospheric pressure at the surface ($y_1 = 0$) and (14.3) becomes

$$p = p_0 + \rho g h \tag{14.4}$$

and it is a constant in the static fluid for given h.

We can apply the above analysis to a small change in the vertical position dy. Then the change in pressure will be $p + dp$ can be written as

$$p + dp = p - \rho g dy$$

where dy.

$$\frac{dp}{dy} = -\rho g \tag{14.5}$$

If ρ and g are constants then

$$dp = -\rho g dy$$

Integrating both parts

$$\int_{p_0}^{p} dp = -\rho g \int_{y_1}^{y_2} dy \qquad \text{then} \qquad p - p_0 = \rho g(y_1 - y_2) = \rho g h$$

and we come to equation (14.3) (remember that $h > 0$ for depth).

*14.2.2 Air pressure with height**

For gases (compressed fluids) the relation between pressure and depth (or altitude) is more complicated because the gas density changes with pressure and temperature. Let us consider a simplified model for air pressure as a function of height. Assume that air density is proportional to pressure or $\rho = \alpha p$ where α is some constant. Then we can rewrite (14.5) as $dp = \alpha p g dy$. Note that we keep dy positive for height. Then equation becomes

$$\frac{dp}{p} = \alpha g dy$$

Integrating both sides

$$\int_{p_0}^{p} \frac{dp}{p} = \alpha g \int_{y_0}^{y} dy \qquad \text{gives} \qquad \ln p - \ln p_0 = \alpha g(y - y_0)$$

or

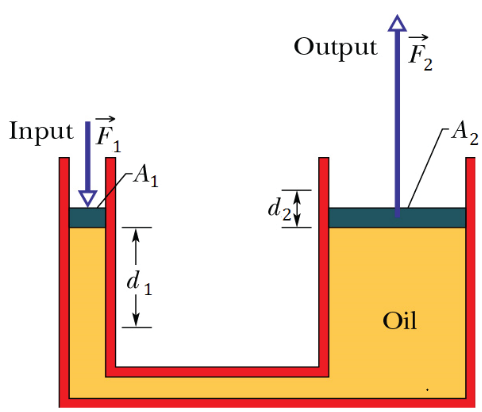
$$p = p_0 e^{-\alpha g h} \tag{14.6}$$

where $h = y - y_0$.

14.2.3 Pascal's principle

From equation (14.4) follows that if we increase the pressure p_0 at the top surface, then the pressure p at any depth increases by exactly the same amount. *A change in the pressure applied to an enclosed incompressible fluid is transmitted undiminished to every point of the fluid and to the walls of its container.* This fact was recognized by Blaise Pascal in 17th century.

There is a very interesting and practical application based on this principle. Let us consider a hydraulic lever.



Assume we applied a force of magnitude F_1 to a piston of surface area A_1 thus increasing pressure in the liquid as $p = F_1/A_1$. The pressure is transmitted through a liquid to another piston of surface area A_2. Because the pressure must be the same on both sides $p = F_1/A_1 = F_2/A_2$ and then

$$F_2 = F_1 \frac{A_2}{A_1} \tag{14.7}$$

Equation (14.7) shows that the output force F_2 must be greater than the input force if $A_2 > A_1$. When we move the input piston downward by a distance d_1, the output piston moves upward by a distance d_2 so that the same volume of the incompressed liquid is displaced $V = A_1 d_1 = A_2 d_2$

$$d_2 = d_1 \frac{A_1}{A_2}$$

and it demonstrate that if $A_2 > A_1$ then the output piston moves a smaller distance. We can also evaluate the work done on the input piston and the work done by the output piston on a load

$$W = F_2 d_2 = \left(F_1 \frac{A_2}{A_1}\right)\left(d_1 \frac{A_1}{A_2}\right) = F_1 d_1$$

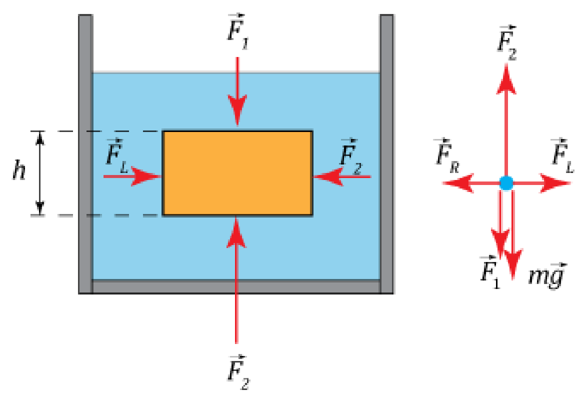
which shows that they are equal. We do not get any energy advantage (sure we cannot!) but we have a force-multiplying device. The number of applications for hydraulic levers are enormous.

14.2.4 Archimedes' principle

Some 2000 years ago Archimedes stated that the magnitude of the buoyant force always equals the weight of the fluid displaced by the object. The buoyant force acts vertically upward through the the center of gravity of the displaced fluid.

We can derive Archimedes' principle using Newton's laws. In equilibrium (hydrostatic) the net force acting on a portion of a fluid must be zero.

Let us consider an object insider a fluid.



The downward force on the top surface is

$$F_1 = p_1 A$$

and the upward force on the bottom surface is

$$F_2 = p_2 A$$

The buoyant force on the object if a vector sum of the two forces

$$F_b = F_2 - F_1 = (p_2 - p_1)A$$

Using equation (14.4) we get

$$F_b = \left(p_1 + \rho_f gh - p_1\right)A = \rho_f ghA = \rho_f gV_f = m_f g$$

or the buoyant force on an object in a fluid is

$$F_b = \rho_f V g = m_f g \tag{14.8}$$

where ρ_f is the density of a fluid, and $m_f = \rho_f V$ is the mass of a fluid. This result is true for an object with arbitrary shape (the proof is a bit longer).

For a body to float the buoyant force must exceed the force of gravity acting on that body.

Case 1: A completely submerged body.

Let us consider a body of mass m and density ρ that is completely submerged into a fluid. Then the net vertical force acting on the body

$$F_{net} = F_b - mg = \rho_f V g - \rho V g = (\rho_f - \rho)V g$$

Thus, for the body to float in a fluid its density must be less or equal to the density of the fluid $\rho \leq \rho_f$. If the density of the object is greater than the density of the fluid, then the buoyant force is less than the downward force of gravity, and the object sinks.

If we place an object on a scale and measure its weight then we have $w = mg$. But is we do the same inside a fluid than the reading is

$$w_{app} = w - F_b$$

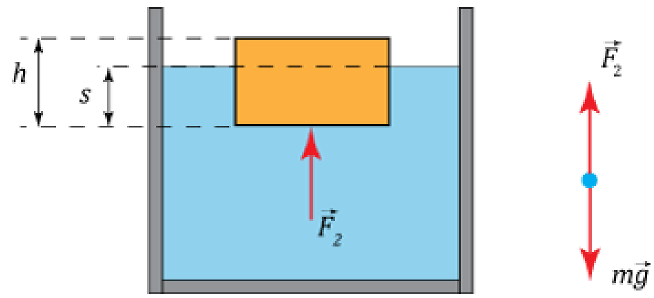
that is called an *apparent weight*.

$$w_{app} = mg - \rho_f V g = mg - \rho_f \frac{m}{\rho} g = mg\left(1 - \frac{\rho_f}{\rho}\right)$$

Note that air has density different from zero (otherwise we would not be able to breath). Therefore there is the buoyant force acting on all objects, including human bodies. Let us quickly estimate such force assuming a volume of a human body to be 0.7m^3. With the air density 1.21 kg/m^3 we get $F_b = 8.3\ N$ or about 1.9 lb.

Case 2: A partly submerged body.

In this case a body floats at a surface of a fluid, or the body is only partially submerged.



Assume that $V_f = sA$ is the volume of the body submerged into a fluid. Then the buoyant force on the body is

$$F_b = pA = \rho_f g s A = \rho_f g V_f$$

The force of gravity on the body is

$$F_g = \rho V g$$

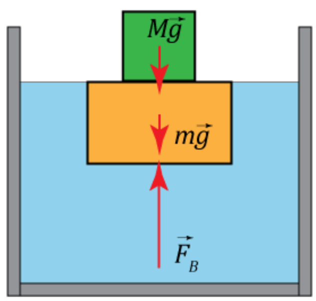
For the body to float we need at least $F_b = F_g$. This we can find the volume of the body that is submerged

$$V_f = V\frac{\rho}{\rho_f} \tag{14.9}$$

It is instructive to evaluate what fraction of an iceberg is submerged. For the density of seawater we have $\rho_w = 0.998 \times 10^3\ kg/m^3$, and for ice $\rho_{ice} = 0.917 \times 10^3\ kg/m^3$, then more than 90% of an iceberg is hidden under water. Now you see the meaning of expression "tip of the iceberg".

Case 3: Maximum load on a floating object.

It is interesting to evaluate the maximum possible load M to place on a floating object just to keep it floating. Assume that the object has a volume of V_o and density ρ_o.



The buoyancy force in this case is

$$F_b = \rho_f V_O g$$

The whole gravitational force

$$F_g = Mg + m_o g = Mg + \rho_o V_o g$$

From the balance of forces it follows

$$M = (\rho_f - \rho_o)V_O$$

where ρ_f is the density of the fluid.

14.2.5 Center of buoyancy and stability

There are two requirements for a body to be in equilibrium, namely 1) the vector sum of all the external forces that act on the body, must be zero, and 2) the vector sum of all the external torques that act on the body, measured about ANY possible point, must be zero.

So far, using simple geometries, our analysis was centered on the net vertical forces acting upon a body. It is time to analyze the net torque created by gravity and the force of buoyancy.

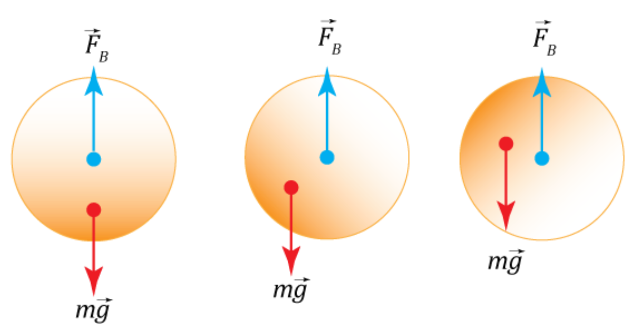
The gravitational force on an extended body is the vector sum of the gravitational forces acting on the individual elements (the atoms) of the body. In section 11.2 we proved that the center of gravity is located at the center of mass as long as the object is in a uniform gravitational field.

The net buoyant force on a body immersed in fluid is also a vector sum of buoyant forces acting on elementary ($\Delta A \rightarrow 0$) surfaces. It is possible to show (not in this course) that the center of buoyancy (the point at which the net buoyancy force acts on the body) is equivalently the geometric center of the submerged portion of the body. Or the center of buoyancy of an object is located at the point that would be the center of mass of the displaced fluid.

Case 1: A completely submerged body.

For a completely submerged body of a uniform density its center of gravity is the same as the center of buoyancy. Then the net torque is zero for any orientation of the body. We have neutral equilibrium.

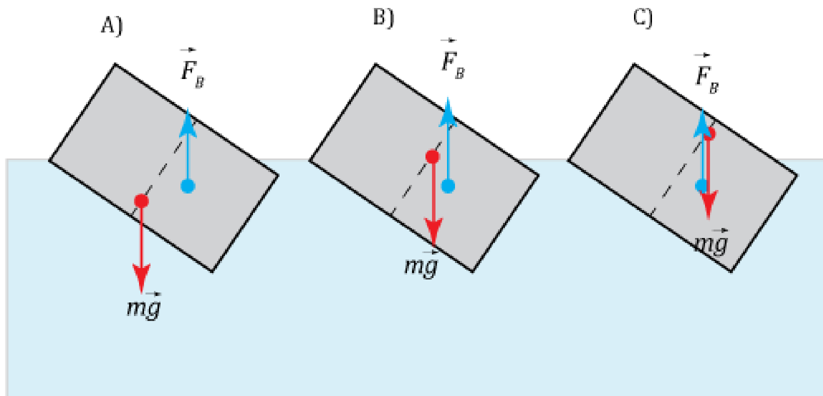
However, if a body does not have uniform density then the outcome depends on relative locations of the two centers. Let us consider an object of a spherical shape when its center of gravity is shifter from the center of the sphere



If the center of gravity is located below the center of buoyancy, then after a small rotation the net torque (around any point) works as a "restoring" torque by bringing the body back to its original position of stable equilibrium. However if the center of gravity is above the center of buoyancy, then a small deviation from the original position results in a net torque that rotates the body from its unstable equilibrium to a stable position (that is different from the initial).

Case 2: A partly submerged body.

For a floating body the situation is more complicated since the center of buoyancy usually shifts when the body is rotated. Such a shift depends on the shape of the body and the orientation in which it is floating.



Understanding physics of stability of floating objects is of paramount importance for engineering, like shipbuilding.

On the figure we consider three situations for a partly submerged object (imagine it is a ship)

For the first situation (the center of gravity is below the center of buoyancy) a deviation from the initial positions results in torques rotating the ship back (we have stable equilibrium here). For B) and C) the center of gravity is located above the center of buoyancy. However, for B) situations, the horizontal shift of the force of buoyancy is larger than for the force of gravity. Therefore the ship will be back to its original upright position. For C) situation the center of gravity is so high that the center of buoyancy is not shifted enough to restore the original position thus both torques rotate the ship in the same direction resulting in capsizing of the ship.

Applying the same analysis to a pencil floating in water we can understand why pencils float only horizontally unless we attach a weight to one of the ends.

14.3 Hydrodynamics

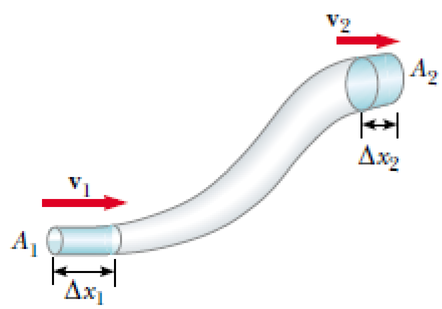
Modelling motion of fluids is one of most challenged problems in physics, including computational challenges. To describe the motion of a fluid we need to describe its properties at every point. First, at different points fluid is moving with different *velocities*. It means we must know the velocity vector (the three components) at every point and for any time. However, the velocity is not the only property that which varies from point to point. We need to deal with the variation of the pressure from point to point. There may also be a variation of density from point to point. So the number of fields needed to describe the complete situation will depend on how complicated the problem is.

In this section we are going to consider motion of a fluid at a lower level of complexity. We will focus on the motion of an *ideal fluid*.

1. We will reduce the complexity of our work by making the assumption that the density is a constant – or the fluid is essentially *incompressible*. Putting it another way, we are supposing that the variations of pressure are so small that the changes in density produced thereby are negligible.
2. The *flow is steady (laminar)*, such that the velocity of the fluid at each point does not change with time.
3. We suppose that the *viscosity of the liquid is unimportant*. Viscosity characterizes internal friction. This is a very strong approximation and it has almost nothing to do with real fluids. John von Neumann characterized the theorist who made such analyses as a man who studied "dry water". However, even such idealized picture leads to understanding of some basic principles of motion of fluids. Motion of fluids with viscosity is a subject of upper level courses.

14.3.1 Conservation of mass

Let us consider a portion of a tube with variable cross section. For a common tube, fluid cannot created or destroyed within the tube, and it cannot be stored. If a mass $M$of the fluid enters through area A_1 during some time interval, then the same mass leaves through area A_2 during the same time interval.



Since the volume can be written as $V = Adx = Avdt$ where v is the fluid's speed, we get for the conservation of mass

$$M = \rho_1 A_1 v_1 dt = \rho_2 A_2 v_2 dt$$

or

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 = \text{a constant} \tag{14.10}$$

The expression above states that the mass flow rate per unit time is constant

For uncompressible fluids (like water) $\rho_1 = \rho_2$ and then

$$A_1 v_1 = A_2 v_2 = \text{a constant} \tag{14.11}$$

or the volume flow rate is constant.

14.3.2 Bernoulli's equation

Normally Bernoulli's equation is derived from a more general equation of motion of a fluid. Since it involves using vector calculus we take an easier approach assuming from the beginning a steady flow of a fluid. By steady flow we mean that at any one place in the fluid the velocity never changes. The fluid at any point is always replaced by new fluid moving in exactly the same way. The velocity picture always looks the same or $\vec{v}$ is a static vector field. A steady flow does not mean that nothing is happening – atoms in the fluid are moving and changing their velocities. It only means that

$$\partial\vec{v}/\partial t = 0$$

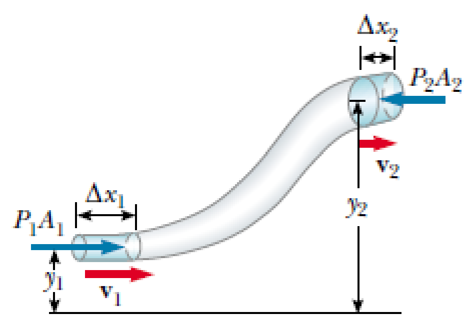
Then we can choose two areas in a non-uniform pipe where the speed of fluid is v_1 in the first are and v_2 in the second area do not change with time.

Now we are ready to apply energy consideration. The force exerted by the fluid is pA . The work done by this force in a time t is $pA\Delta x = p\Delta V$, where ΔV is the volume of the corresponding section. The work done by pressure on fluid in the first section is positive

$$W_1 = p_1\Delta V$$

but the work in the second section id negative because the fluid force opposes the displacement

$$W_2 = -p_2\Delta V$$



Considering pressure as a cause of an external force that changes kinetic and gravitational potential energy of the fluid we can write

$$\Delta K + \Delta U_g = W_1 + W_2$$

Assuming that the mass of fluid is conserved, Δm is our at the first area and Δm is in at the second area

$$\frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 + \Delta m g y_2 - \Delta m g y_1 = p_1\Delta V - p_2\Delta V$$

Using $\Delta m = \rho\Delta V$ and rearranging the terms yields

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 = \text{a constant} \tag{14.12}$$

This is Bernoulli's theorem. The constant may in general be different for different flows; all we know is that it is the same all along a given flow. The Bernoulli's principle is in fact nothing more than a statement of the conservation of energy. A conservation theorem such as this gives us a lot of information about a flow without having to solve the detailed equations of motion.

Attentions: Bernoulli's principle is valid only for incompressible and steady flow fluids with no internal friction (no viscosity).

You can quickly test the Bernoulli principle by using two pieces of paper holding them close together and trying to blow them apart. They actually come together. The reason, of course, is that the air has a higher speed going through the constricted space between the sheets than it does when it gets outside. The pressure between the sheets is lower than atmospheric pressure, so they come together rather than separating.

14.4 Examples

* force on a vertical wall (calculus – integration)

*** see more in my handwritten notes

*** problems on ice in a glass of water, melting ice.

* water flowing from a water tank (find speed) F2-40-7

A friend asks you how much pressure is in your car tires. You know that the tire manufacturer recommends 30 psi, but it's been a while since you've checked. You can't find a tire gauge in the car, but you do find the owner's manual and a ruler. Fortunately, you've just finished taking physics, so you tell your friend, "I don't know, but I can figure it out." From the owner's manual you find that the car's mass is 1500 kg. It seems reasonable to assume that each tire supports one-fourth of the weight. With the ruler you find that the tires are 15 cm wide and the flattened segment of the tire in contact with the road is 13 cm long. What answer will you give your friend?

How many helium-filled toy balloons would be required to lift you? Take the size of an average helium toy balloon to be 40 cm in diameter.

The three stooges (Moe, Larry, and Curly) make a log raft by lashing together oak logs of diameter 0.3 m and length 1.8 m. The total weight of the stooges is 480 lb.

a) How many logs will be needed to keep them afloat in fresh water?
b) What fraction of the raft will be above the water surface without the stooges on the raft?
c) Curly says that there is no point to make the raft since, as he read in a magazine, the average density of a human body is less than the density of water. They would float anyway. Do you agree or disagree with Curly? Provide a very good and supported answer.

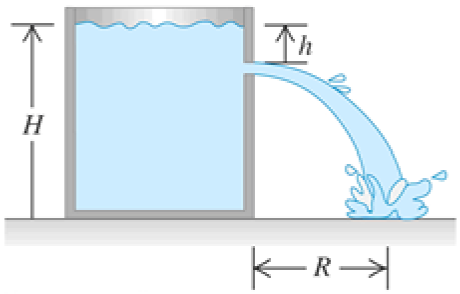
Reference data for density: fresh water 1.0 g/cm3, wood (oak) 0.8 g/cm3.

A hurricane wind blows across a 6.0 m * 15.0 m flat roof at a speed of 130 km/h.

a) Is the air pressure above the roof higher, or lower than the pressure inside the house? Explain
b) What is the pressure difference?
c) How much force is exerted on the roof? If the roof cannot withstand this much force, will it "blows in" or "blows out"?

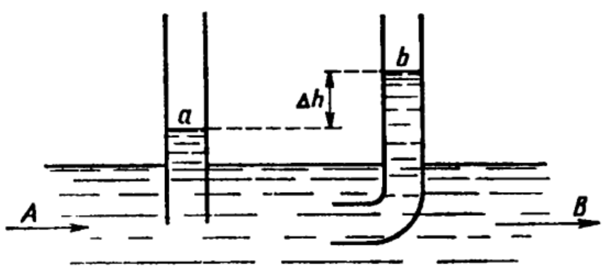
Water stands at a depth H in a large, open tank whose side walls are vertical. A hole is made in one of the walls at a depth h below the water surface.

a) At what distance R from the foot of the wall does the emerging stream strike the floor
b) At what depth should the hole be placed to make the emerging stream strike the ground at the maximum distance from the base of the tank?



Also find time to have water out of the tank

4.8. A liquid flows along a horizontal pipe AB (Fig. 6). The difference between the levels of the liquid in tubes a and b is 10 cm. The diameters of tubes a and b are the same. Determine the velocity of the liquid flowing along pipe AB.



Answer: 1.4 m/s.

15 Waves

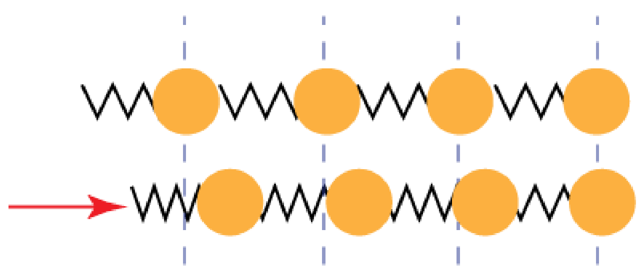
Most likely when you hear a word 'weaves' you think about water waves. If you had physics classes before you may have some knowledge about sound waves, or even electromagnetic waves.

15.1 Mechanical waves (physics behind the scene)

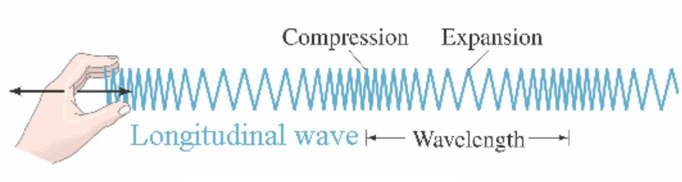
In this chapter we are going to talk about mechanical waves that propagate in some physical medium (liquid, gas, solid body). And sure the first question – what physics is behind mechanical waves?

In a physical medium atoms and/or molecules interact with their neighbors. Therefore, if we create a perturbation in one point of physical medium, such perturbation propagates through the medium.

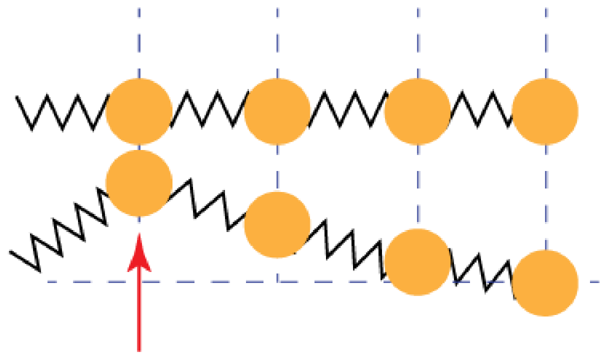
Imagine a long chain of balls connected by springs. If we provide a short perturbation (a kick) to a spring on the left, then this perturbation will propagate in time from one ball to another.



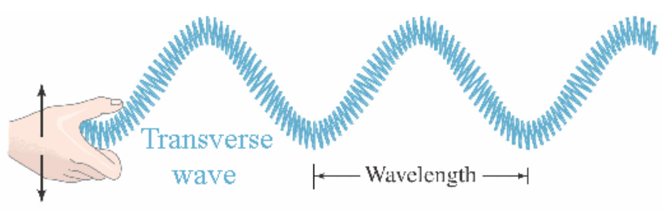
We may reasonably guess that the speed of such perturbation is linked to the spring constants (the larger the spring constant the faster the perturbation propagates) and to the mass of the blocks (the larger the mass, the slower the speed of the propagation). Replacing balls and springs on atoms provides a simple model that represents what is called *longitudinal waves*, when a traveling wave causes the particles of the medium to move parallel to the direction of wave motion. Quite often a continuous spring serves as a good model for longitudinal waves.



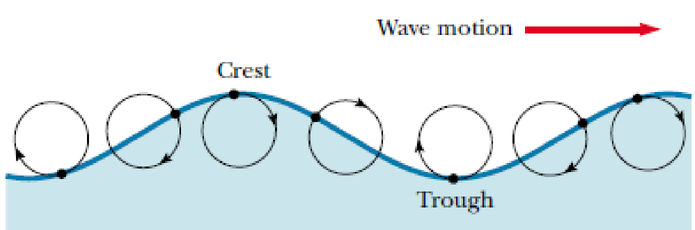
Now we kick one of the masses in a vertical direction. Then a travelling perturbation causes the masses to move perpendicular to the wave motion.



Such wave is called a *transverse wave*. It is also possible to well model such a wave with a continuous spring as



There are waves that are a combination *of transverse and longitudinal displacements*. For example, water molecules on the surface of water, in which a wave is propagating, move in nearly circular paths.



Each molecule is displaced both horizontally and vertically from its equilibrium position.

It is important to note that in every of these waves each atom/molecule oscillates about its equilibrium point. Therefore a wave is a propagation of a perturbation (propagation of energy), not matter.

15.2 Wave equation*

For describing a motion of a particle we use Newton's second that provides a position as a function of time $\vec{r}(t)$. A wave is not localized in space but is a function of a position and time. For example for a transverse wave in the above example we need $y(x,t)$ to describe such a wave.

Let us analyze a wave on a string under tension. Such example allows us apply Newton's second law in a straightforward way. This analysis goes a bit beyond requirements for standard university physics course. However it is very instructive to see how much can we derive from Newton's second law.

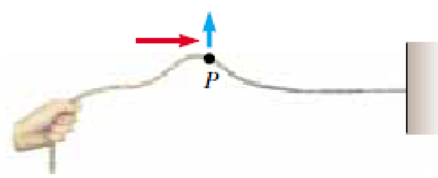
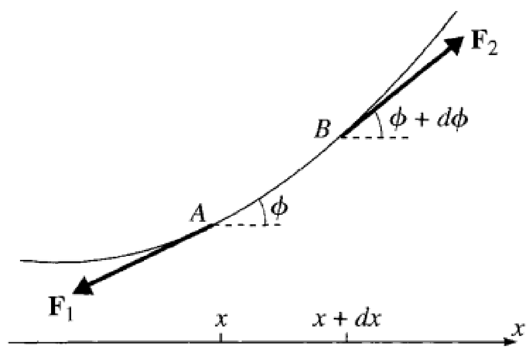


Figure below shows a rope displaced transversely from its equilibrium position along the horizontal x −axis. In equilibrium, we assume that the string lies exactly on the x axis.



To explore the motion of the string, we shall apply Newton's second law to a small segment AB of the string, between x and $x + dx$. To simplify our discussion we shall ignore gravity, and we shall assume that the displacement $y(x,t)$ remains so small for all x and all t, that the string remains nearly parallel to the x axis. This guarantees that the string's length is essentially unchanged and hence that the tension T remains the same for all x and all t.

Let us now consider the net force $\vec{F} = \vec{F}_1 + \vec{F}_2$ acting on the element of the rope. The horizontal component of this force is

$$F_{x,net} = T\cos(\phi + d\phi) - T\cos\phi = -T\sin\phi\, d\phi$$

where T is the magnitude of the tension in the rope. Since ϕ and $\phi + d\phi$ are both small, this is doubly small, and $F_{x,net}$ is negligible, consistent with our assumption that the motion is in the y direction only.

The y component is

$$F_{y,net} = T\sin(\phi + d\phi) - T\sin\phi$$

That we can write (do you see the derivative of the sine function here?)

$$F_{y,net} = T\cos\phi\, d\phi$$

Certainly that the y component is not negligible. Since ϕ is small we can replace $\cos\phi$ by 1, and we can write $d\phi = (\partial\phi/\partial x)dx$. The derivative is a partial derivative since $\phi(x,t)$ depends on x and t. Finally, again since ϕ is small, $\phi = \partial y/\partial x$, the slope of the string. Therefore,

$$F_{y,net} = Td\phi = T\frac{\partial\phi}{\partial x}dx = T\frac{\partial^2 y}{\partial x^2}dx.$$

By newton's second law, $F_{y,net} = ma_y$, where a_y is the acceleration

$$a_y = \frac{\partial^2 y}{\partial t^2}$$

and mass is the mass of the segment AB, equal to μdl where μ is the linear density (mass per unit length) and dl is the length of the segment AB. For small angles ϕ we have $dx = \cos\phi\, dl = dl$. Thus

$$F_{y,net} = \mu\frac{\partial^2 y}{\partial t^2}dx$$

Equating the two expressions for the force we get for the equation of the string

$$\frac{\partial^2 y}{\partial t^2} = \left(\frac{T}{\mu}\right)\frac{\partial^2 y}{\partial x^2} \tag{15.1}$$

Introducing the notation

$$v = \sqrt{\frac{T}{\mu}} \tag{15.2}$$

where T is the tension in our string and μ is its linear mass density. Then we can rewrite (15.1) as

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \tag{15.3}$$

The equation of motion (15.3) is called the one-dimensional wave equation since its solutions are waves traveling along the string. It is a partial differential equation, involving derivatives with respect to x and t. The constant v has the dimensions of speed

$$\left[\frac{\text{Tension}}{\mu}\right] = \frac{[ML/T^2]}{[M/L]} = \left[\frac{L^2}{T^2}\right] = \text{speed}^2$$

and it is the speed with which the waves travel. Here we just note that this quantity depends on internal properties of the medium, in this case the tension and mass density of a rope. The wave equation (15.3) governs the motion of many different waves.

For waves on strings, y represents the vertical displacement of the string. For sound waves, y corresponds to displacement of air molecules from equilibrium or variations in either the pressure or the density of the gas through which the sound waves are propagating.

Analysis of the wave equations (15.3) requires a good knowledge of partial differential equation. In this chapter we are going to bring an important result following from (15.3) without proof.

Assume that at initial time $t = 0$ the shape of the pulse can be represented as $y = f(x)$. Then measured in a stationary reference frame the wave function (that is the solution of the wave equation) is

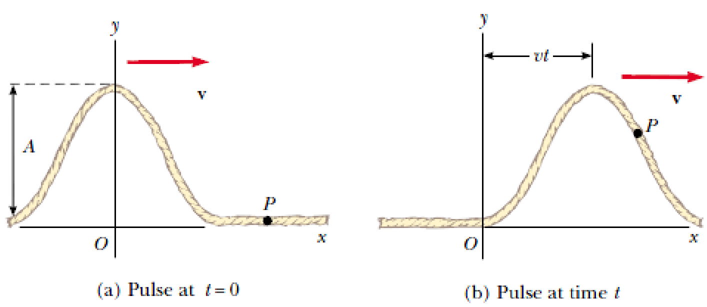
$$y = f(x - vt)$$

for the pulse travelling to the right.

If the wave pulse travels to the left, the string displacement is

$$y = f(x + vt)$$

For any given time t, the wave function y as a function of x defines a curve representing the shape of the pulse at this time.



Hence, the wave speed is $v = dx/dt$.

Attention: The wave equation (15.3) describes propagation of a wave in space and time. Note that for finding a unique solution for the wave we need both initial conditions (shape of the pulse or wave at $t = 0$) and boundary conditions (real strings are finite in lengths and have ends).

15.3 Sinusoidal waves

An important special case of the solution (15.3) is the case that the function f is sinusoidal. The sinusoidal wave is the simplest example of a periodic continuous wave and can be used to build more complex waves.

Assume that at time $t = 0$ the wave function can be written as

$$y(x, t=0) = A \sin\left(\frac{2\pi}{\lambda}x\right) \tag{15.4}$$

where the constant A represents the wave amplitude and the constant λ is the wavelength. If the wave moves to the right with a speed v, then the wave function at some later time t is

$$y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]. \tag{15.5}$$

By definition, the wave travels a distance of one wavelength in one period T. Therefore, the wave speed, wavelength, and period are related by the expression

$$v = \frac{\lambda}{T}$$

Substituting this into (15.5) gives (15.6)

$$y(x,t) = A \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

This form of the wave function shows the periodic nature of y. At any given time t (a snapshot of the wave), y has the same value at the positions $x, x+\lambda, x+2\lambda$, and so on. Furthermore, at any given position x, the value of y is the same at times $t, t+T, t+2T$, and so on.

We can express the wave function in a convenient form by defining the angular wave number k

$$k = \frac{2\pi}{\lambda} \tag{15.7}$$

and the angular frequency

$$\omega = \frac{2\pi}{T} \tag{15.8}$$

Then we can write equation (15.6) in a standard form

$$y(x,t) = A \sin(kx - \omega t) \tag{15.9}$$

If the wave were traveling to the left, we replace $x - vt$ by $x + vt$ in (15.5) or $kx - \omega t$ by $kx + \omega t$ in equation (15.9).

If the vertical displacement y is not equal to zero at $t = 0$, then we can express the wave function with a phase shift

$$y(x,t) = A \sin(kx - \omega t + \varphi)$$

where φ is the phase constant. This constant can be determined from the initial conditions.

Using the definitions for k, ω and frequency $f = 1/T$ we can write v in two more forms

$$v = \lambda f = \frac{\omega}{k}.$$

We can show that the sinusoidal wave function (15.9) is a solution of the wave equation (15.3). The appropriate derivatives are

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(kx - \omega t)$$

Substituting these expressions into (15.3) gives

$$-\omega^2 = -v^2 k^2$$

And since $v = \omega/k$ we have an identity.

15.4 Power transferred by a wave

While a wave does not transfer matter, it does transfer energy. We will proceed with a model of a wave on a string. Power is defined as

$$P = \vec{F} \cdot \vec{v}$$

In the case of the string we can write

$$P = F_y v_y = -T \sin\phi \, \frac{\partial y}{\partial t}$$

where T is the tension in the string. Since we work with small angles

$$\sin\phi \approx \tan\phi = \frac{\partial y}{\partial x}$$

Then

$$p = -T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$$

Calculating derivatives of the sinusoidal wave (15.9) gives

$$P = -T[kA\cos(kx - \omega t)][-\omega A \cos(kx - \omega t)] = T\omega k A^2 \cos^2(kx - \omega t)$$

Earlier we got that $v = \sqrt{T/\mu}$ or $T = v^2\mu$. Using also $k = \omega/v$ we get

$$P = v^2 \mu\omega \frac{\omega}{v} A^2 \cos^2(kx - \omega t) = \mu v \omega^2 A^2 \cos^2(kx - \omega t).$$

The average power at any location x is then

$$P_{av} = \frac{1}{2}\mu v \omega^2 A^2 \tag{15.10}$$

because the average value of $\cos^2(kx - \omega t)$ is $1/2$. This average is taken over an entire period of the motion with x held constant. This shows that the rate of energy transfer by a sinusoidal wave on a string is proportional to the wave speed, the square of the frequency, and the square of the amplitude.

15.4.1 Wave intensity

We live in a three-dimensional world. Most types of waves (sound waves, seismic waves) carry energy across all three dimensions. For such waves the *intensity* is a common characteristic that describes energy transported by the wave in unit time, per unit area, across a surface perpendicular to the direction of propagation. That is, intensity I is average power per unit area. If a point source emits waves uniformly in all directions, then the energy at a distance from the source is distributed uniformly on a spherical surface of radius r and area $A = 4\pi r^2$ and the intensity is

$$I = \frac{P_{av}}{4\pi r^2} \tag{15.11}$$

The intensity of a three-dimensional wave varies inversely with the square of the distance from a point source. It is usually measured in watts per square meter.

15.5 Interference and reflection of waves

15.5.1 Superposition and interference

More often we encounter situations whet many waves are travelling through a medium. To analyze such wave combinations, we use of the superposition principle. The principle states that if two or more traveling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves. Waves that obey this principle are called linear waves and are generally characterized by small amplitudes. Waves that violate the superposition principle are called nonlinear waves and are often characterized by large amplitudes.

Attention: One consequence of the superposition principle is that two traveling waves can pass through each other without being destroyed or even altered.

The combination of separate waves in the same region of space to produce a resultant wave is called interference. For two waves on a string we can then write

$$y(x,t) = y_1(x,t) + y_2(x,t) \tag{15.12}$$

Mathematically, this additive property of wave functions follows from the form of the wave equation (15.3). Specifically, the wave equation is linear; that is, it contains the function only to the first power. As a result, if any two functions and satisfy the wave equation separately, their sum also satisfies it and is therefore a physically possible motion. Because this principle depends on the linearity of the wave equation and the corresponding linear-combination property of its solutions, it is also called the principle of linear superposition.

When two waves cause increase in a resulting displacement then such superposition is called as constructive interference. When the displacements caused by the two waves are in opposite directions, we call such superposition as destructive interference.

There are very many interesting effects and applications of wave interference.

15.5.2 Reflection of waves

Let us consider a traveling wave when it encounters a change in the medium. For example a string attached to a rigid support at one end. When the wave reaches the support, the wave undergoes reflection that is, the wave moves back along the string in the opposite direction. And the reflected wave is inverted.

If a wave reaches a boundary between two mediums then part of the incident wave is reflected and part undergoes transmission, or some of the wave passes through the boundary. For example, suppose a light string is attached to a heavier string. When a wave traveling on the light string reaches the boundary between the two, part of the wave is reflected and inverted and part is transmitted to the heavier string. The reflected pulse is inverted as in the case of the string rigidly attached to a support. The following general rules apply to reflected waves: When a wave travels from medium A to medium B and $v_A > v_B$ (that is, when B is denser than A), the wave is inverted upon reflection. When a wave travels from medium A to medium B and $v_A < v_B$ (that is, when A is denser than B), the wave is not inverted upon reflection.

zzz: work more with this part (using the wave equation?)

15.5.3 Standing waves in a string

Let us look at a sinusoidal wave reflected by a fixed end of a string. If we continue shaking one end of a string, a travelling wave will be reflected from the other end. Both the travelling and reflected waves will interfere with each other. Usually it will be quite a noise of oscillations. However, if we vibrate the string at one of right frequencies the two waves will interfere in such a way that a standing wave will be formed. If the length of the string is L then the condition for standing waves is

$$\lambda_n = \frac{2L}{n}, \quad n = 1,2,3 \ldots$$

where λ is the wavelength. In order to find the frequency f of such vibrations we use $f = v/\lambda$. Then

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} = nf_1$$

where f_1 is the fundamental frequency.

zzz: add a figure + talk about nodes (and energy does not pass beyond nodes) +math representation

15.6 Sound waves

Sound waves are likely the most important of longitudinal waves in our lives. Sound waves may be described as variations of pressure at various points. We still can employ the wave equation (15.3) but in this case the speed of sound waves depends on the compressibility and inertia of the medium. If the medium has a bulk modulus B (bulk modulus measures the resistance of medium to changes in their volume) and density ρ, then the speed of sound waves in fluids can be written as

$$v = \sqrt{B/\rho} \tag{15.13}$$

Note that the speed of sound also depends on the temperature of the medium. It is good to know that the speed of sound in air is approximately 343 m/s at temperature 20^0C.

The speed of sound in a solid material can be found from

$$v = \sqrt{Y/\rho} \tag{15.14}$$

where Y is Young's modulus,

Mathematically a sinusoidal sound wave can be represented as the pressure variation

$$\Delta P = \Delta P_0 \sin(kx - \omega t)$$

where ΔP_0 is the pressure amplitude, which is the maximum change in pressure from the equilibrium value.

15.6.1 Intensity of sound; Decibels

The intensity of a wave, or the power per unit area, to be the rate at which the energy being transported by the wave flows through a unit area A perpendicular to the direction of travel of the wave. The human ear can detect a very wide range of intensities. For this reason it is convenient to use a logarithmic scale, where the sound level β is defined by the equation

$$\beta \text{ (in dB)} = 10 \log \frac{I}{I_0} \tag{15.15}$$

where I_0 is the intensity of some reference level taken to be at the "threshold of hearing" which is $I_0 = 1.0 \times 10^{-12}\, W/m^2$. Thus, for example, the sound level of a sound intensity $I = 1.0 \times 10^{-5}\, W/m^2$ will be

$$\beta = 10 \log \frac{1.0 \times 10^{-5}\, W/m^2}{1.0 \times 10^{-12}\, W/m^2} = 10 \log 10^7 = 70\, dB$$

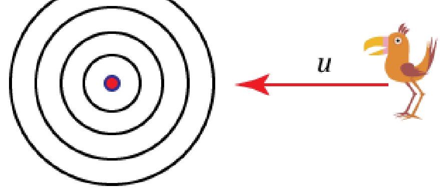
that corresponds to a sound level of a busy street.

Note that the threshold of hearing corresponds to 0 dB.

15.6.2 Doppler Effect

When a source of sound is moving the frequency of the sound we hear is different than when the source is at rest. This phenomenon is called the Doppler effect and occurs for all types of waves (not only sound waves).

Case 1: Moving observer



Consider a source of sound with the frequency f, the wavelength λ, and the speed of sound to be $v = \lambda f$. If the observer were also stationary, the observer would detect f wave fronts per second. (That is, when and the observed frequency equals the source frequency.) When the observer moves toward the source with the speed u, the speed of the waves relative to the observer is $v + u$ but the wavelength λ is unchanged. Hence, the frequency heard by the observer is increased and is given by

$$f' = \frac{v+u}{\lambda} = f\frac{v+u}{v} = f\left(1+\frac{u}{v}\right)$$

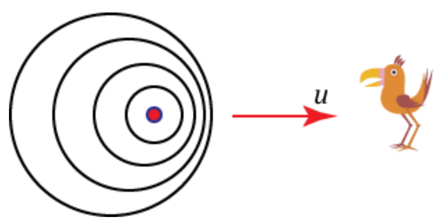
Thus is the observer is moving toward source the detected frequency increases

$$f' = f\left(1+\frac{u}{v}\right) \tag{15.16}$$

If the observer is moving away from the source, then the speed of the wave relative to the observer is decreased as $v - u$ and the frequency heard by the observer in this case is also decreased and is given by

$$f = f\left(1-\frac{u}{v}\right) \tag{15.17}$$

Case 2: Moving source



Now consider the case when the source is in motion and the observer is stationary. If the source moves directly toward observer then the wave fronts heard by the observer are closer together than they would be if the source were not moving. As a result, the wavelength λ' measured by observer is shorter than the wavelength λ of the source. During each vibration, which lasts for a time T, the source moves a distance $uT = u/f$ and the wavelength is

$$\lambda' = \lambda - \frac{u}{f}$$

Then the frequency heard by observer is

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - \frac{u}{f}} = \frac{v}{\frac{v}{f} - \frac{u}{f}} = f\frac{1}{\left(1-\frac{u}{v}\right)}$$

For the source moving away from observer we get

$$f' = f\frac{1}{\left(1+\frac{u}{v}\right)}$$

If both source and observer are in motion, we find the following general equation for the observed frequency:

$$f' = \left(\frac{v \pm u_O}{v \mp u_S}\right) f \tag{15.18}$$

where the upper signs apply to motion of one toward the other, and the lower signs apply to motion of one away from the other.

zzz More subjects: Musical instruments, interference of sound waves, beats, shock waves