PHYS 3010: Classical Mechanics

Lecture Notes

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Contents

Chapter 1

Introduction

1.1 Prelude and Overview

What is Classical Mechanics (CM)?

CM deals with the *motion* of *material objects* through *space and time* and with the *laws* that govern that motion.

Analysis:

Let's go through this definition step by step:

(i) *Material objects*: The central property of the objects of CM is (inertial) mass. We will see soon that mass is to be interpreted as resistance to acceleration.
An important concept is that of a *mass point*: $\Leftrightarrow$ a pointlike particle with mass as its only property. Certainly, the idea of a (classical) mass point is an idealization, but it is a useful one for two reasons: it greatly simplifies actual calculations and it often is a good approximation even if the objects are not small. Think of the Solar System. What makes the idea of viewing the planets and the Sun as mass points work is that the distances between them are large compared to their sizes.

(ii) *Motion through space and time*: There are a few (operational) things to be said about space and time (pertaining to this course).

Space: is three-dimensional and Euclidian.

$\rightarrow$ One can define Cartesian coordinate systems, and the mathematical description of what happens in space is facilitated by vector algebra in $\mathbb{R}^3$.

Time: is just a homogeneous parameter.

$\rightarrow$Motion: an object passes (continuously) through different positions as time goes by, i.e., motion can be characterized by a trajectory $\mathbf{r}(t)$ and its derivatives:

$$\left.\begin{array}{l} \bullet \text{ trajectory } \mathbf{r}(t) \\[1em] \bullet \text{ velocity } \mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \dot{\mathbf{r}}(t) \\[1em] \bullet \text{ acceleration } \mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) \end{array}\right\} \begin{array}{l} \Rightarrow \text{ `kinematics':} \\ \text{mathematical description} \\ \text{of motion without} \\ \text{consideration of causes (forces)} \end{array}$$

Extensions of the notions of space and time (beyond this course)

- special theory of relativity $\rightarrow$ 4-dimensional "Minkowski" space (space-time)
- general theory of relativity $\rightarrow$ (locally) curved space
- quantum mechanics $\rightarrow$ ∞-dimensional Hilbert space

(iii) *The governing laws*: So, what causes the motion of objects? The answer—according to Newton—is: forces. Analyzing the motion of objects subject to forces is called dynamics (in contrast to kinematics, where forces are not considered). This analysis is built on Newton's Laws, which we will recap shortly.

But first, let's ask: What are we going to do in PHYS 3010 given that Newton's Laws are studied in PHYS 1010 and 2010?
The answer is: We will develop and apply alternate formulations of CM, the so-called Lagrangian and Hamiltonan formulations (which are equivalent to Newton's). Why?

- To gain insight into the foundations of CM and physics in general. For instance, we will discuss Hamilton's principle which is an example of a variational principle. Variational principles are in use in other branches of physics, e.g. in quantum mechanics.

- To learn a powerful problem-solving technique (i.e., the application of the Lagrangian Equations of Motion).

1.2 Recap of Newtonian Mechanics

1.2.1 Newton's Laws (1687)

Lex prima: "Every body continues in its state of rest, or uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it."

Lex secunda: "The change of motion is proportional to the motive force impressed and is made in the direction of the line in which that force is impressed."

Lex tertia: "To every action there is always imposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts."

In addition, Newton formulated a corollary which is sometimes called his fourth law. Its content is the *principle of superposition of forces*, i.e., it states that forces add like vectors and that it is the *net force* that causes the change of motion of an object according to his second law. Moreover, he gave a definition of the 'motion' referred to in that law. It is the (linear) momentum

$$\mathbf{p} = m\mathbf{v}$$

with the mass m and the velocity $\mathbf{v}$.

Analysis

- The first law is Galileo's principle of inertia. It includes the insight that the states of rest and uniform motion are equivalent. As a consequence, the descriptions of physical processes from the perspective of two reference frames which move uniformly with respect to each other are also equivalent. In other words, this postulate introduces, somewhat implicitly, inertial reference frames and Galilean transformations.

Definition of an **inertial reference frame**: a reference frame in which a forcefree body moves uniformly or is at rest (i.e., a system in which Newton's first law holds).

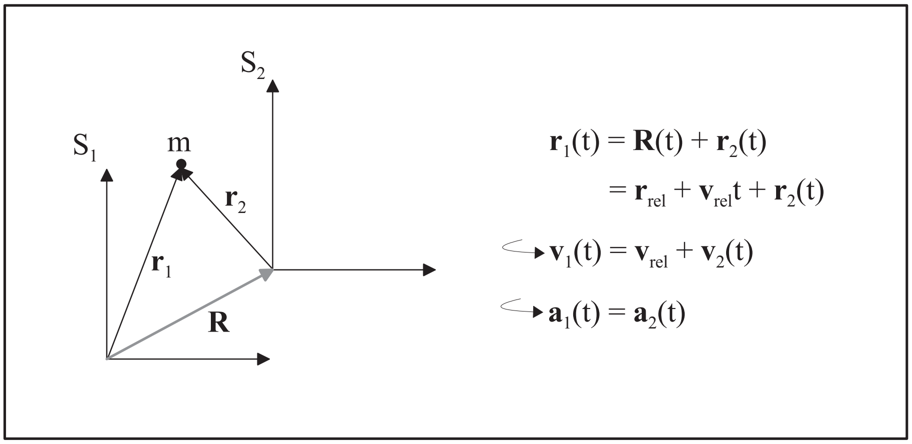


Figure 1.1: Mass point m seen from two reference frames which move uniformly ($\mathbf{R}(t) = \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}t$) with respect to each other.

According to the first law we have:

$$\begin{array}{lllll} \text{if} & \mathbf{F} = 0 & \longrightarrow & \mathbf{a_1} = \mathbf{a_2} = 0 \\ \text{if} & \mathbf{F} \neq 0 & \longrightarrow & \mathbf{a_1} = \mathbf{a_2} \neq 0 \end{array}$$

- The second law tells us (quantitatively) what happens if one or more than one forces act on a body. It is the fundamental equation of motion (EoM) of CM:

$$\boxed{\dot{\mathbf{p}} = \mathbf{F}_{\text{net}} \equiv \sum_i \mathbf{F}_i}$$

$$\Longleftrightarrow \quad \frac{d}{dt}(m\mathbf{v}) = \mathbf{F}_{\text{net}}$$

if $\dot{m} = 0$:

$$\boxed{m\dot{\mathbf{v}} = m\mathbf{a} = m\ddot{\mathbf{r}} = \mathbf{F}_{\text{net}}}$$

$\dot{m} = 0$ is certainly fulfilled for mass points, but $\dot{m} \neq 0$ is possible, too: Think of a rocket whose fuel is burnt. In more general terms (and beyond the scope of this course) Einstein showed us that $\dot{m} \neq 0$ is actually not the exception but the rule, because the mass of a moving object depends on its speed, and therefore (in general) on time.

Consequences:

(i) If $0 = \mathbf{F}_{\text{net}} = \dot{\mathbf{p}} \Longrightarrow \mathbf{p} = \text{const} \longrightarrow$ recover first law!

(ii) Invariance of EoM wrt. Galilean transformations (see above):

$$S_1 : \mathbf{F_1} = m\mathbf{a_1} = m\mathbf{a_2} = \mathbf{F_2} : S_2$$

This is why inertial reference frames are so important: the forces are the same in all of them, hence they are all equivalent for the description of a physical process.

- The third law expresses a fundamental property of physical forces[1]: action equals reaction. Let's consider two interacting mass points:

$$\left.\begin{aligned} \mathbf{F}_{12} &: \text{ force on particle 1 due to particle 2} \\ \mathbf{F}_{21} &: \text{ force on particle 2 due to particle 1} \end{aligned}\right\} \mathbf{F}_{12} = -\mathbf{F}_{21}$$

$$\xrightarrow{2^{\text{nd}}\text{law}} \quad m_1\mathbf{a}_1 = \mathbf{F}_{12} = -\mathbf{F}_{21} = -m_2\mathbf{a}_2$$

$$\Longrightarrow \quad \frac{m_1}{m_2} = \frac{|\mathbf{a}_2|}{|\mathbf{a}_1|} \equiv \frac{a_2}{a_1}$$

If one fixes the absolute scale by introducing a standard mass (the kilogram) the last equation expresses a dynamical definition of mass in

[1] which, according to our current understanding of the fundamental interactions, has to be formulated somewhat differently in modern physics.

terms of accelerations: the smaller mass speeds up faster. Hence, we can interpret mass as the resistance of an object to acceleration.

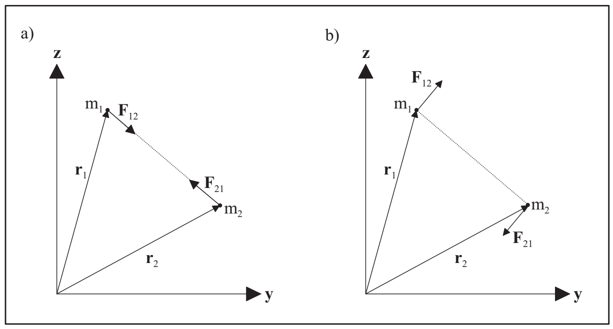


Figure 1.2: Illustrations of the third law: the force vectors can be, but are not necessarily directed along the line that joins the particles.

Note that Newton's third law is fulfilled for the gravitational and the electric forces, but not (at least not directly) for magnetic forces.

<u>Further comments</u>

(i) The physical origin of forces is (normally) not discussed in CM.

(ii) Instead, the basic problem of CM is to solve Newton's EoM for given forces. The EoM (normally) is a second-order ordinary differential equation (ODE) of the form

$$\ddot{\mathbf{r}}(t) = \mathbf{f}(\mathbf{r}, \dot{\mathbf{r}}, t) \qquad \text{(+ initial conditions)}$$

Unfortunately, analytical solutions are only known for a few cases, but numerical procedures are readily available (see, e.g., the various computer problems in [FC]).

(iii) Conservation laws—briefly reviewed in the next section—emerge as consequences of Newton's Laws.

1.2.2 (Linear) momentum, angular momentum, work, and energy

a) Momentum

simplest situation: one forcefree mass point (MP):
$\mathbf{F} = 0 \quad \Longrightarrow \quad \mathbf{p} = m\mathbf{v}_0 = \text{const}$ momentum conservation

$$\hookrightarrow \quad \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t \qquad \text{uniform motion}$$

$\sphericalangle$ system of N MPs and some useful notions:

- 'internal force' $\mathbf{f}_{ki}$: (force exerted on MP k by MP i)
- 'external force' $\mathbf{F}_k$: external force on kth MP
- 'isolated system': a system without external forces ($\mathbf{F}_k = 0$ for $k = 1, ..., N$)
- total mass: $M = \sum_{k=1}^{N} m_k$
- centre of mass $\mathbf{R} = \frac{1}{M} \sum_{k=1}^{N} m_k \mathbf{r}_k$
- centre-of-mass velocity: $\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{M} \sum_{k=1}^{N} m_k \mathbf{v}_k$
- total = centre-of-mass momentum: $\mathbf{P} = M\mathbf{V} = \sum_{k=1}^{N} m_k \mathbf{v}_k = \sum_{k=1}^{N} \mathbf{p}_k$
- position of a MP wrt. centre of mass: $\mathbf{r'}_k = \mathbf{r}_k - \mathbf{R}$

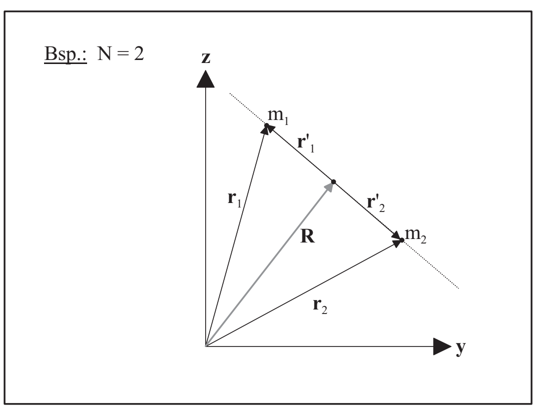


Figure 1.3: Definition of the centre of mass.

$\triangleleft$ EoM of kth MP:

$$
\begin{array}{rrcl}
 & \dot{\mathbf{p}}_k & = & \mathbf{F}_k + \sum_{i=1}^{N} \mathbf{f}_{ki} \\
 \text{note that} & \mathbf{f}_{ki} & = & -\mathbf{f}_{ik} \quad \longrightarrow \quad \mathbf{f}_{kk} = 0 \\
\longrightarrow & \sum_{k=1}^{N} \dot{\mathbf{p}}_k & = & \sum_{k=1}^{N} \mathbf{F}_k + \sum_{i,k=1}^{N} \mathbf{f}_{ki} \\
 & \shortparallel & & \shortparallel \qquad\quad \shortparallel \qquad (3^{\text{rd}}\ \text{law})
\end{array}
$$

$$
\boxed{\dot{\mathbf{P}} \quad = \quad \mathbf{F}_{ext} \quad + \quad 0}
$$

The centre of mass of a particle system moves as if it were a single particle of mass M acted on by the total external force.

Momentum conservation (holds in isolated systems):

$$
\boxed{\text{if} \quad \mathbf{F}_{ext} = 0 \quad \Longrightarrow \quad \dot{\mathbf{P}} = 0 \quad \Longrightarrow \quad \mathbf{P} = \text{const}}
$$

This makes the centre of mass a convenient choice for the origin of an inertial reference frame (for an isolated system).

b) Angular momentum

definition for one particle:

$$
\begin{aligned}
\mathbf{l} &= \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) \\
|\mathbf{l}| &= l = rp \sin\gamma
\end{aligned}
$$

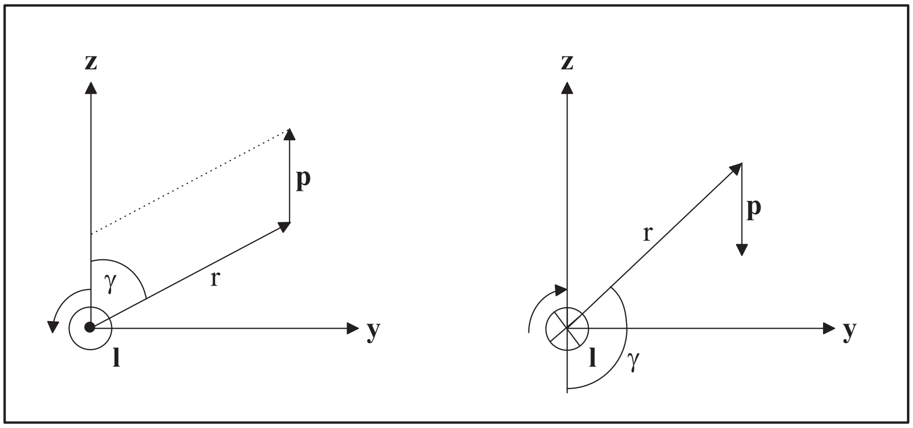


Figure 1.4: On the left panel the angular momentum vector points out of the yz-plane, on the right panel it points into it.

$\triangleleft$ $$\dot{\mathbf{l}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = m(\mathbf{v} \times \mathbf{v}) + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F}$$

<u>definition:</u> torque $\mathbf{N} = \mathbf{r} \times \mathbf{F}$

$$\longrightarrow \quad \boxed{\dot{\mathbf{l}} = \mathbf{N}}$$

$$\boxed{\text{if} \quad \mathbf{N} = 0 \quad \Longrightarrow \quad \dot{\mathbf{l}} = 0 \quad \Longrightarrow \quad \mathbf{l} = \text{const}}$$

The former equation is the fundamental EoM for rotational motion. The latter expresses angular momentum conservation.

$$\begin{array}{llll} \mathbf{N} = 0 \quad \text{if} & (i) & \mathbf{F} = 0 & \\ & (ii) & \mathbf{F} \parallel \mathbf{r} & \text{central force} \end{array}$$

Let's consider a system of N MPs. The total angular momentum is defined as

$$\begin{aligned}
\mathbf{L}(t) &= \sum_{k=1}^{N} \mathbf{l}_k(t) \\
&= \sum_{k} \big(\mathbf{r}_k(t) \times \mathbf{p}_k(t)\big) \\
\longrightarrow \dot{\mathbf{L}} &= \sum_{k} (\mathbf{r}_k \times \dot{\mathbf{p}}_k) \\
&= \sum_{k=1}^{N} (\mathbf{r}_k \times \mathbf{F}_k) + \sum_{i,k=1}^{N} (\mathbf{r}_k \times \mathbf{f}_{ki})
\end{aligned}$$

$$\begin{aligned}
\lhd : \quad \sum_{i,k} (\mathbf{r}_k \times \mathbf{f}_{ki}) &= \sum_{i,k} (\mathbf{r}_i \times \mathbf{f}_{ik}) \;=\; \frac{1}{2} \sum_{i,k} \big\{(\mathbf{r}_k \times \mathbf{f}_{ki}) + (\mathbf{r}_i \times \mathbf{f}_{ik})\big\} \\
&= \frac{1}{2} \sum_{i,k} \big\{(\mathbf{r}_k \times \mathbf{f}_{ki}) - (\mathbf{r}_i \times \mathbf{f}_{ki})\big\} \\
&= \frac{1}{2} \sum_{i,k} \big\{(\mathbf{r}_k - \mathbf{r}_i) \times \mathbf{f}_{ki}\big\} \\
&= 0 \quad \text{if} \quad \mathbf{f}_{ki} \parallel (\mathbf{r}_k - \mathbf{r}_i)
\end{aligned}$$

total torque is defined as

$$\mathbf{N} = \sum_{k} (\mathbf{r}_k \times \mathbf{F}_k) = \sum_{k} \mathbf{N}_k$$

Accordingly, we have derived

$$\boxed{\dot{\mathbf{L}} = \mathbf{N}}$$

$$\boxed{\text{if} \quad \mathbf{N} = 0 \quad \Longrightarrow \quad \dot{\mathbf{L}} = 0 \quad \Longrightarrow \quad \mathbf{L} = \text{const}}$$

In particular, total angular momentum is conserved in isolated systems.

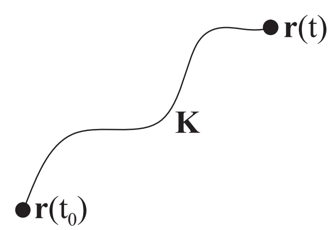
c) Work and energy

Let's start again with one MP. The most general definition of work is as follows: if $\mathbf{r}(t)$ is the path on which the MP travels in the time interval $[t_0,\ t]$ the associated work W is

$$W = \int_{t_0}^{t} \mathbf{F}\big(\mathbf{r}(t'),\ \mathbf{v}(t'),\ t'\big) \cdot \mathbf{v}(t')\ dt'.$$

If the force is a vector field, i.e., if $\mathbf{F} = \mathbf{F}(\mathbf{r})$ W is given by a line integral

$$\begin{aligned} W &= \int_K \mathbf{F}(\mathbf{r})\ d\mathbf{r} \qquad \text{with} \\ d\mathbf{r} &= \mathbf{v}(t')\ dt' \end{aligned}$$



Discussion:

(i) For uniform $\mathbf{F}$ and rectilinear motion: $W = \mathbf{F} \cdot \mathbf{r}$

(ii) $W = 0$ if $\mathbf{F} \perp d\mathbf{r}$

Example 1: If you simply hold (but do not move) a mass m in Earth's gravity field $\longrightarrow$ $W = 0$ (since $\mathbf{v} = 0$)

Example 2: uniform circular motion

$$\begin{aligned} \mathbf{r} &= (R\cos\omega t,\ R\sin\omega t) \\ \mathbf{v} &= (-R\omega\sin\omega t,\ R\omega\cos\omega t) \\ \mathbf{a} &= (-R\omega^2\cos\omega t,\ -R\omega^2\sin\omega t) \\ &= -\omega^2\mathbf{r} \\ \longrightarrow m\mathbf{a} &= \mathbf{F}(\mathbf{r}) \\ &= -m\omega^2\mathbf{r} \\ \longrightarrow\ \mathbf{F}\cdot\mathbf{v} = 0 &\iff \mathbf{F} \perp d\mathbf{r} = \mathbf{v}\ dt \implies W = 0 \end{aligned}$$

Let's connect work with Newton's EoM:

$$\triangleleft \qquad \begin{aligned} W &= \int_{t_0}^{t} \mathbf{F}\big(\mathbf{r}(t'), \mathbf{v}(t'), t'\big) \cdot \mathbf{v}(t')\ dt' \\ &\overset{\text{EoM}}{=} m \int_{t_0}^{t} \dot{\mathbf{v}}(t') \cdot \mathbf{v}(t')\ dt' \\ &= \frac{m}{2} \int_{t_0}^{t} \frac{d}{dt'}\big(\mathbf{v}^2(t')\big)\ dt' \\ &= \frac{m}{2}\big(\mathbf{v}^2(t) - \mathbf{v}^2(t_0)\big) \\ &= \frac{1}{2m}\big(\mathbf{p}^2(t) - \mathbf{p}^2(t_0)\big) \end{aligned}$$

<u>definition:</u> kinetic energy

$$T = \frac{m}{2}\mathbf{v}^2 = \frac{\mathbf{p}^2}{2m} \geq 0$$

$$\boxed{\longrightarrow \quad T(t) = T(t_0) + W(t_0 \to t)} \quad \text{'work-energy theorem'}$$

The work-energy theorem holds for all forces. A stronger relation is obtained if the force field is *conservative*. For a conservative force a scalar potential energy function U exists such that

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$$

Note that U is defined only up to a constant by this equation. Normally one fixes this constant by requiring $U(\mathbf{r}) \overset{\mathbf{r}\to\infty}{\longrightarrow} 0$. The conservativity of a force can be formulated in different (but equivalent) ways:

$$\boxed{\begin{array}{ccc} \mathbf{F} = -\nabla U & \Longleftrightarrow & \nabla \times \mathbf{F} = 0 \\ \Updownarrow & \nwarrow\searrow & \Updownarrow \\ \displaystyle\int_1^2 \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} & \Longleftrightarrow & \displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \\ \text{independent of path} & & \end{array}}$$

$$\begin{aligned} \text{work} - \text{energy theorem} \;\longrightarrow\; W = U(1) - U(2) &= T(2) - T(1) \\ \Longleftrightarrow \quad T(1) + U(1) &= T(2) + U(2) \end{aligned}$$

$$\boxed{E = T + U = \text{const}} \quad \text{conservation of energy}$$

In a more general situation the total force might consist of conservative and dissipative parts

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_{conservative} + \mathbf{F}_{dissipative} \\ \nabla \times \mathbf{F} &= \nabla \times \mathbf{F}_{diss} \neq 0 \end{aligned}$$

$\triangleleft$ Newton's EoM:

$$\begin{aligned} m\dot{\mathbf{v}} &= -\nabla U(\mathbf{r}) + \mathbf{F}_{diss} \qquad \Big| \cdot \mathbf{v} \\ \mathbf{F}_{diss} \cdot \mathbf{v} &= m\dot{\mathbf{v}}\mathbf{v} + \nabla U \mathbf{v} \\ \Longleftrightarrow \mathbf{F}_{diss} \cdot \mathbf{v} &= \frac{d}{dt}\Big(\frac{m}{2}\mathbf{v}^2 + U(\mathbf{r}(t))\Big) \end{aligned}$$

$$\Longleftrightarrow \quad \boxed{\frac{d}{dt}\Big(T + U\Big) = \frac{dE}{dt} = \mathbf{F}_{diss} \cdot \mathbf{v}}$$

energy is not conserved in this case, but changes according to the power associated with the dissipative force.

Let's consider a conservative N-particle system. Starting from Newton's EoM $\dot{\mathbf{p}}_k = \mathbf{F}_k + \sum_{i=1}^{N} \mathbf{f}_{ki}$ one obtains (see Appendix A.1)

$$\boxed{\begin{aligned} \frac{d}{dt}\Big(T + U\Big) &= 0 \\ T + U = E = \text{const} & \end{aligned}} \quad \text{energy conservation}$$

where

- $T = \sum_k T_k = \sum_k \frac{m_k}{2}\mathbf{v}_k^2$: total kinetic energy

- $U = \sum_k U_k + \sum_{k<i} \bar{U}_{ki} = \sum_k U_k + \frac{1}{2} \sum_{k \neq i} \bar{U}_{ki}$: total potental energy (note that $\bar{U}_{ki} = \bar{U}_{ik}$)

if the following applies:

(i) $\mathbf{F}_k = \mathbf{F}_k(\mathbf{r}_k) = -\nabla_k U_k(\mathbf{r}_k)$ (conservative external forces)

(ii) $\nabla_k \times \mathbf{f}_{ki} = \nabla_i \times \mathbf{f}_{ik} = 0$ (conservative internal forces)

(iii) $\mathbf{f}_{ki} = \mathbf{f}_{ki}(\mathbf{r}_k - \mathbf{r}_i) = -\mathbf{f}_{ik}(\mathbf{r}_k - \mathbf{r}_i)$ (internal forces fulfill Newton's third law and depend only on the relative coordinate $\mathbf{r}_k - \mathbf{r}_i$).

Chapter 2

Hamilton's Principle — Lagrangian and Hamiltonian dynamics

2.1 Preliminary formulation of Hamilton's principle (1834/35)

Of all the possible paths along which a particle may move from one point to another in a given time interval $[t_1, t_2]$ the actual path followed is that which minimizes the integral

$$S = \int_{t_1}^{t_2} (T - U)\ dt$$

<u>Comments:</u>

(i) $L := T - U$: Lagrangian (function)
obviously, the Lagrangian has the dimension of an energy, but it is not the energy of the system. Note that the definition implies that the system under study is conservative.

(ii) $S = \int_{t_1}^{t_2} L\, dt$: 'action integral'

(iii) Hamilton's principle (HP) is an integral principle (in contrast to Newton's law of motion)

(iv) HP is a fundamental principle of modern physics.

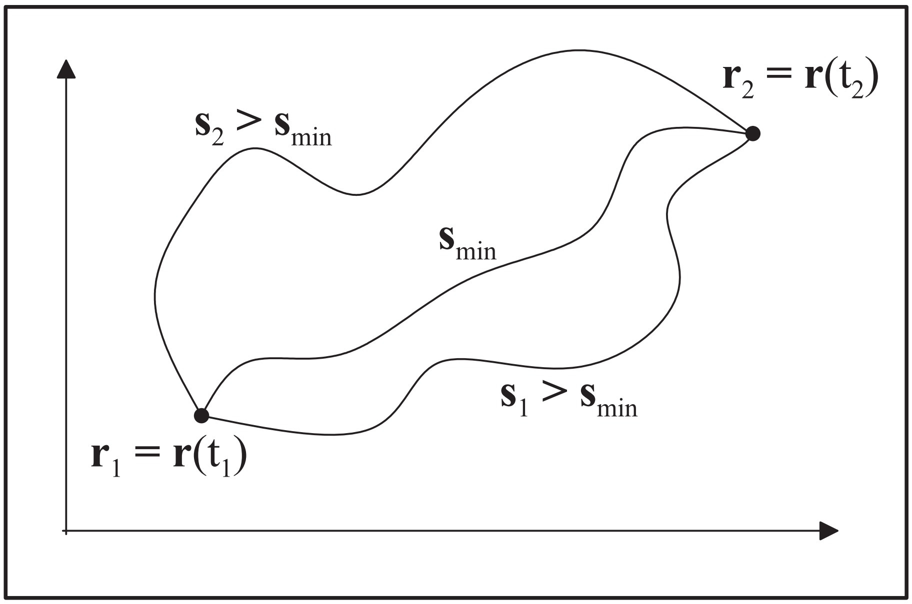


Figure 2.1: Illustration of Hamilton's principle

(v) EoMs can be derived from HP
in order to do this we need a few basics of the calculus of variations.

2.1.1 Calculus of variations

given: reasonably well-behaved function $f(x, \dot{x}, t)$

sought-after: 'path' $x(t)$ with $x(t_1) = x_1$ and $x(t_2) = x_2$ such that

$$I = \int_{t_1}^{t_2} f(x, \dot{x}, t)\, dt \qquad \text{assumes extremum value}$$

<u>Theorem</u>: a necessary condition for I to assume an extremum value for $x(t)$ is Euler's equation:

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0}$$

<u>Proof</u>: assume that $x(t)$ is the sought-after path
define neighbouring paths by

$$x(\alpha, t) = x(t) + \alpha\eta(t) \qquad \text{with } \eta(t_1) = \eta(t_2) = 0$$

$$\hookrightarrow \qquad \dot{x}(\alpha, t) = \dot{x}(t) + \alpha\dot{\eta}(t)$$

the arbitrary function $\eta(t)$ describes the deformation of the path and the (real) parameter α is a scale factor that determines its magnitude

$$\lessdot \qquad \int_{t_1}^{t_2} f(x(\alpha, t), \dot{x}(\alpha, t), t)\ dt = I(\alpha)$$

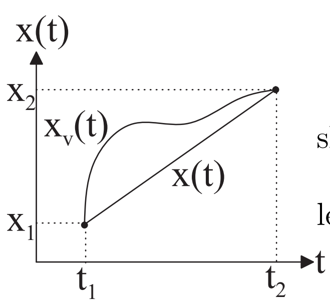
the integral can be viewed as an ordinary function of α. If this function assumes an extremum value at $\alpha = 0$ (i.e., for $x = x(0, t)$) it follows that

$$\frac{dI}{d\alpha}\Big|_{\alpha=0} = 0$$

let's work out the derivative:

$$
\begin{aligned}
\frac{dI}{d\alpha} &= \frac{d}{d\alpha}\int_{t_1}^{t_2} f(x(\alpha,t),\dot{x}(\alpha,t),t)\ dt \\
&= \int_{t_1}^{t_2} \frac{\partial}{\partial\alpha} f(x(\alpha,t),\dot{x}(\alpha,t),t)\ dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\alpha} + \frac{\partial f}{\partial\dot{x}}\frac{\partial\dot{x}}{\partial\alpha}\right)\ dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}\ \eta(t) + \frac{\partial f}{\partial\dot{x}}\ \dot{\eta}(t)\right)\ dt \\
\text{integration by parts :}\qquad &= \int_{t_1}^{t_2} \frac{\partial f}{\partial x}\ \eta(t)\ dt + \frac{\partial f}{\partial\dot{x}}\ \eta(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\ dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial\dot{x}}\right)\ \eta(t)\ dt \\
&\quad (\text{since } \eta(t_1) = \eta(t_2) = 0) \\
\hookrightarrow \qquad \frac{dI}{d\alpha}\Big|_{\alpha=0} &= 0 \ = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial\dot{x}}\right)\ \eta(t)\ dt \quad \text{for } x = x(0,t) \\
\overset{\eta(t)\text{ arbitrary}}{\Longrightarrow} \qquad & \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial\dot{x}} = 0
\end{aligned}
$$

<u>Elementary example:</u>



shortest distance between two points in a plane

length of path: $I = \int_{t_1}^{t_2} \sqrt{1+\dot{x}^2}\ dt$

$$\hookrightarrow \quad f = \sqrt{1+\dot{x}^2} = f(\dot{x})$$

$$\frac{\partial f}{\partial x} = 0\ ; \quad \frac{\partial f}{\partial\dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} \quad \xrightarrow{\text{Euler}} \quad \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0$$

$$\Longrightarrow \quad \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = const = C_1$$

$$\hookrightarrow \quad \dot{x} = \pm\sqrt{\frac{C_1^2}{1-C_1^2}} = C_2 \quad \Longrightarrow \quad x(t) = C_2 t + C_3 \qquad \text{(rectilinear path)}$$

<u>Remarks</u>:

(i) A more interesting (and quite famous) application is the so-called brachistochrone problem. It is example 6.2 in both [Tay] and [TM].

(ii) If the first derivative of a function vanishs the function is said to be *stationary* at that point. Stationarity is necessary, but not sufficient for a a minimum (not even for an extremum). Since the Euler equation is equivalent to the stationarity of I at $\alpha = 0$ we don't know if its solution really minimizes the integral. In fact, there are counterexamples (see, e.g., Chap 6.3 of [Tay]) and, accordingly, a more cautious statement of HP (which will be discussed later) refers only to the stationarity of the action integral. For practical applications, this is of no concern.

(iii) δ-notation
It is quite common to use a short-hand notation in variational calculus—the so-called δ-notation. To introduce it let's go back to the calculation of $\frac{dI}{d\alpha}$ on the previous page. We had

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right) \eta(t)\ dt$$

this can be rewritten by noting that $\eta(t) = \frac{\partial x}{\partial \alpha}$ and multiplying both sides by a 'small' scale factor $d\alpha$

$$\begin{aligned} \frac{dI}{d\alpha} d\alpha &= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right) \frac{\partial x}{\partial \alpha} d\alpha\ dt \\ \Longleftrightarrow \delta I &= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right) \delta x\ dt \end{aligned}$$

with the 'variations' $\delta I = \frac{dI}{d\alpha} d\alpha$ and $\delta x = \frac{\partial x}{\partial \alpha} d\alpha$.

The stationarity of I is then expressed as

$$\begin{aligned} \delta I &= 0 \\ \Longleftrightarrow & \delta \int_{t_1}^{t_2} f(x, \dot{x}, t)\, dt = \int_{t_1}^{t_2} \delta f\, dt \\ &= \int_{t_1}^{t_2} \Big(\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \Big)\, \delta x\, dt = 0 \end{aligned}$$

More about the δ-notation: [TM], Chap. 6.7

(iv) A more rigorous definition of variations δI and δx is based on the mathematical notion of a functional. A functional $I[x]$ is a generalized function: it maps a function to a number: $x(t) \mapsto I$. Our integral is a typical example

$$I[x] = \int_{t_1}^{t_2} f(x.\dot{x}, t)\, dt$$

One can construct a calculus for functionals, i.e., set up precise definitions and rules of how to differentiate them. Functional derivatives then turn out to be closely related to our variations, i.e., the calculus for functionals provides a foundation of the symbolic notation introduced above.

(v) More on calculus of variations: [TM], Chap. 6; [Tay], Chap. 6

2.1.2 HP for a simple case

$\lessdot$ 1 single mass point in one-dimensional world

$$L = T - U = \frac{m}{2}\dot{x}^2 - U(x) = L(x, \dot{x}) \ ; \quad F(x) = -\frac{dU}{dx}$$

$$\text{HP :} \qquad \delta S = \delta \int_{t_1}^{t_2} L(x, \dot{x})\, dt = 0$$

$$\Longleftrightarrow \quad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0} \quad \text{Lagrangian equation}$$

let's work out the derivatives:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \; ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \; ;$$
$$\frac{\partial L}{\partial x} = -\frac{dU}{dx} = F(x)$$

$$\boxed{\hookrightarrow \quad m\ddot{x} = F(x) \quad \Longleftrightarrow \quad \text{Lagrange} \quad \Longleftrightarrow \quad \text{HP}}$$

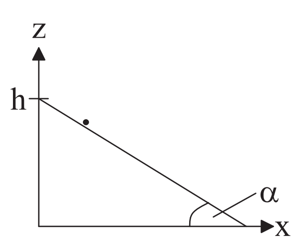
Appendix A.2 deals with the question of whether S always assumes a minimum on the actual path (in the 1D world).

2.2 Constrained systems and generalized coordinates

2.2.1 Preliminary for one mass point

a) Examples for constraints

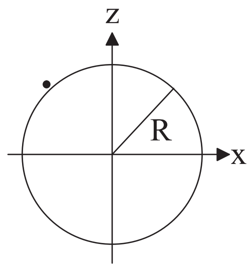
(i) Motion on an inclined plane (without friction)



constraint: $z = (-\tan\alpha)x + h$

$\longrightarrow$ number of degrees of freedom (dofs) is reduced to 2

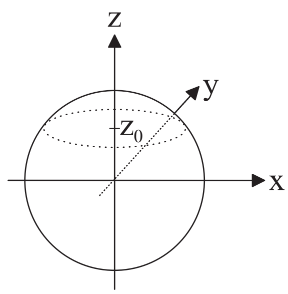
(ii) Motion on the surface of a sphere (no friction)



constraint: $x^2 + y^2 + z^2 = R^2$

$\longrightarrow$ two dofs

(iii) Motion on the rim of a circle of radius $r^2 = R^2 - z_0^2$

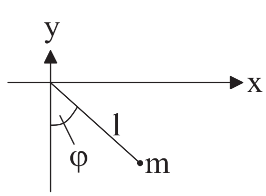


constraints: $x^2 + y^2 + z^2 = R^2$
$z = z_0 < R$
$\longrightarrow$ one dof

special case: $z_0 = R$:
<u>zero</u> dof (particle cannot move)

if z_0 >R the two constraints are not compatible

(iv) Planar pendulum



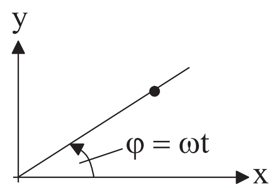
constraints: $z = 0$
$l = \sqrt{x^2 + y^2} = const$

same as example (iii)

all constraints in examples (i) - (iv) are characterized by equations of the form $f(x, y, z) = 0$. They are called *scleronomic-holonomic* constraints.

But there are other types of constraints:

(v) Bead on a uniformly rotating wire

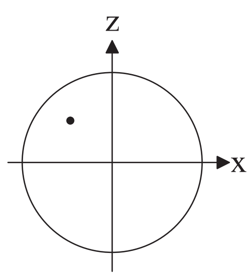


constraints: $y = \tan(\omega t)x$
$z = 0$

$\longrightarrow$ one dof

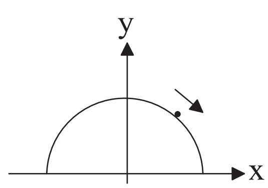
the first constraint is characterized by an equation of the form $f(x, y, z, t) = 0$, i.e., the constraint involves the time. It is called *rheonomic-holonomic*.

(vi) Mass point within a spherical container



constraint: $x^2 + y^2 + z^2 < R$

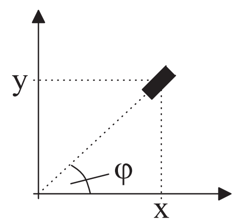
variant:



$x^2 + y^2 + z^2 \geq R^2$

$\longrightarrow$ these are *nonholonomic* constraints (characterized by inequalities)
$\longrightarrow$ number of dofs is *not* reduced.

(vii) Disk rolling on a horizontal plane (without slipping)



bird eye's view (for a special situation)

centre of mass velocity is locked to angular velocity $\omega = \dot{\psi}$. Hence:

$$\begin{aligned} \mathbf{v_{CM}} &= \pm v_{CM} \cos\varphi \hat{\imath} \pm \sin\varphi \hat{\jmath} \\ v_{CM} &= R\dot{\psi} \end{aligned}$$

$\rightarrow$ *differential* (nonholonomic) constraints:

$$dx_{CM} = \pm R d\psi \cos\varphi\ , \qquad dy_{CM} = \pm R d\psi \sin\varphi$$

$\rightarrow$ number of dofs is <u>not</u> reduced

<u>Summary</u>

$$\text{holonomic constraints} \begin{cases} \text{scleronomic}: & f(x,y,z) = 0 \\ \text{rheonomic}: & f(x,y,z,t) = 0 \end{cases}$$

$$\text{nonholonomic constraints} \begin{cases} \text{inequalities} \\ \text{differential} \end{cases}$$

holonomic constraints reduce the number of dofs, but nonholonomic ones do not.

b) Generalized coordinates and Lagrangian equations

$\sphericalangle$ one-particle system with holonomic constraint $f(x,y,z) = 0$. Let's assume that the equation of constraint can be solved for $z = z(x,y)$. Then, the z-coordinate can be eliminated from the Lagrangian:

$$\begin{aligned} \longrightarrow \quad L(x,y,z,\dot{x},\dot{y},\dot{z},t) &= L(x,y,z(x,y),\dot{x},\dot{y},\dot{z}(x,y,\dot{x},\dot{y}),t) \\ &= \tilde{L}(x,y,\dot{x},\dot{y},t) \end{aligned}$$

<u>see example (i)</u>: inclined plane

constraint: $f(x,y,z) = h - x\tan\alpha - z = 0 \Longleftrightarrow z(x) = h - x\tan\alpha$

in addition let's assume $y = 0$ and that gravity acts on the particle:

$$\begin{aligned} T &= \frac{m}{2}\mathbf{v}^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{x}^2 + \dot{x}^2 \tan^2\alpha) \\ U &= mgz = mg(h - x\tan\alpha) \\ \hookrightarrow \quad L = T - U &= \frac{m}{2}\dot{x}^2(1 + \tan^2\alpha) - mg(h - x\tan\alpha) \\ &= L(x, \dot{x}) \end{aligned}$$

$$\text{HP} \quad \Longleftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

work out the derivatives:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}(1 + \tan^2\alpha),\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{m\ddot{x}}{cos^2\alpha},\ \frac{\partial L}{\partial x} = mg\tan\alpha$$

$$\stackrel{\text{Lag.eq.}}{\Longrightarrow} \qquad m\ddot{x} - mg\tan\alpha\cos^2\alpha = 0$$

$$\Longleftrightarrow \qquad \boxed{\ddot{x} = g\sin\alpha\cos\alpha}$$

$$\begin{aligned} \underline{\text{solution}}: \qquad x(t) &= \frac{g}{2}(\sin\alpha\cos\alpha)t^2 + C_1 t + C_2 \\ z(t) &= h - \frac{g}{2}(\sin^2\alpha)t^2 - (C_1 t + C_2)\tan\alpha \end{aligned}$$

One obtains—of course— the same result in the Newtonian framework. The treatment is, however, rather different. One has to recognize that the plane exerts a normal force on the particle such that the resulting (net) force points in the direction of the inclination. The acceleration is then determined by the net force, i.e., the projection of the gravitational force in this direction:

$$ma = F_{\text{net}} = mg\sin\alpha$$

integration yields

$$s(t) = \frac{g}{2}t^2\sin\alpha + C_3 t + C_4$$

this result can be written in terms of the x- and z-coordinates by using $\sin\alpha = (h-z)/s$ and $x = s\cos\alpha$. Apparently, the advantage of the Lagrangian treatment is that it is not necessary to think about the normal force (i.e., the force of constraint).

<u>see example (iv):</u> planar pendulum
let's assume that gravity is acting: $U = mgy$

$$\text{constraints}: \quad \begin{aligned} z &= 0 \\ x &= \pm\sqrt{l^2 - y^2} \end{aligned}$$

$$\begin{aligned}
L = T - U &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy \\
&= \frac{m}{2}\Big(\frac{y^2\dot{y}^2 + \dot{y}^2(l^2 - y^2)}{l^2 - y^2}\Big) - mgy \\
&= \frac{m}{2}\Big(\frac{\dot{y}^2 l^2}{l^2 - y^2}\Big) - mgy = L(y, \dot{y}) \\
\longrightarrow \quad & \text{this looks complicated!}
\end{aligned}$$

It is a much better idea to work in polar coordinates:

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} = l = const \\
\tan\varphi &= -\frac{x}{y} \\
\Longleftrightarrow \quad x &= r\sin\varphi \qquad\qquad y = -r\cos\varphi \\
&= l\sin\varphi \qquad\qquad\;\;\; = -l\cos\varphi
\end{aligned}$$

$$\begin{aligned}
U &= mgy = -mgl\cos\varphi \\
\dot{x} &= l\dot{\varphi}\cos\varphi\ , \qquad \dot{y} = l\dot{\varphi}\sin\varphi \\
\hookrightarrow \quad L &= \frac{m}{2}l^2\dot{\varphi}^2 + mgl\cos\varphi = L(\varphi, \dot{\varphi})
\end{aligned}$$

Lagrangian equation of motion:

$$\frac{\partial L}{\partial \dot{\varphi}} = ml^2\dot{\varphi}\ , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = ml^2\ddot{\varphi}\ , \quad \frac{\partial L}{\partial \varphi} = -mgl\sin\varphi$$

$$\Longrightarrow \quad ml^2\ddot{\varphi} + mgl\sin\varphi = 0$$

$$\boxed{\ddot{\varphi} + \omega^2 \sin\varphi = 0 \; , \quad \omega = \sqrt{\frac{g}{l}}}$$

compare this to the Newtonian treatment (where another force of constraint, the tension of the rod, must be introduced).

For small angles one can assume $\sin\varphi \approx \varphi$

$$\Longrightarrow \quad \ddot{\varphi} + \omega^2 \varphi = 0 \quad \Longrightarrow \quad \varphi(t) = a\sin(\omega t - \beta)$$

for larger angles the oscillation is nonlinear and the solution much more involved (see [TM], Chap. 4.4).

another example: (a variant of) the cycloidal pendulum
a cycloid is the path of a point on the edge of a circular wheel that roles on a straight line. If the wheel roles in x-direction the equations of the cycloid are

$$\begin{aligned} x &= R(\alpha - \sin\alpha) \\ y &= R(1 - \cos\alpha) \end{aligned}$$

where α is the angle through which the rolling circle rotates.

Let's consider a bead that slides on a wire that has the form of an *inverted* cycloid

$$\begin{aligned} x &= R(\alpha - \sin\alpha) \\ y &= R(1 + \cos\alpha) \end{aligned}$$

The suitable (generalized) coordinate to express the Lagrangian is the angle α. The potential energy is

$$U = mgy = mgR(1 + \cos\alpha)$$

we need

$$\begin{aligned} \dot{x} &= R\dot{\alpha}(1 - \cos\alpha) \\ \dot{y} &= -R\dot{\alpha}\sin\alpha \end{aligned}$$

to rewrite the kinetic energy

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = mR^2\dot{\alpha}^2(1 - \cos\alpha)$$

$$\hookrightarrow L = T - U = mR^2\dot{\alpha}^2(1 - \cos\alpha) - mgR(1 + \cos\alpha) = L(\alpha, \dot{\alpha})$$

derivatives:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\alpha}} &= 2mR^2\dot{\alpha}(1 - \cos\alpha) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} &= 2mR^2(\ddot{\alpha}(1 - \cos\alpha) + \dot{\alpha}^2 \sin\alpha) \\
\frac{\partial L}{\partial \alpha} &= mR^2\dot{\alpha}^2 \sin\alpha + mgR\sin\alpha
\end{aligned}$$

Lagrangian equation of motion

$$\Longrightarrow 2\ddot{\alpha}(1 - \cos\alpha) + \dot{\alpha}^2 \sin\alpha - \frac{g}{R}\sin\alpha = 0$$

this seemingly complicated equation of motion can be simplified by an ingenious coordinate transformation:

step (i): use $1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$, $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$\hookrightarrow 2\ddot{\alpha}\sin\frac{\alpha}{2} + \dot{\alpha}^2\cos\frac{\alpha}{2} - \frac{g}{R}\cos\frac{\alpha}{2} = 0$$

step (ii): substitute $u = -\cos\frac{\alpha}{2}$:

$$\begin{aligned}
\hookrightarrow \dot{u} &= \frac{\dot{\alpha}}{2}\sin\frac{\alpha}{2} \\
\ddot{u} &= \frac{\ddot{\alpha}}{2}\sin\frac{\alpha}{2} + \frac{\dot{\alpha}^2}{4}\cos\frac{\alpha}{2} \\
&= \frac{1}{4}(2\ddot{\alpha}\sin\frac{\alpha}{2} + \dot{\alpha}^2\cos\frac{\alpha}{2}) \\
&\xrightarrow{\text{EoM}} \quad \ddot{u} + \frac{g}{4R}u = 0
\end{aligned}$$

this is nothing but the equation of motion of an harmonic oscillator of frequency $\omega = \sqrt{\frac{g}{4R}}$. The motion is said to be *isochronous*, i.e., it has a frequency that is independent of the amplitude. To complete the solution one has to find $\alpha(t)$ by using the solution $u(t)$ in the coordinate transformation, and then insert $\alpha(t)$ in the cycloid equations to find $x(t)$ and $y(t)$ (try it!).

Conclusions:

- Holonomic constraints can be used to eliminate coordinates.
- Suitable coordinates yield simple Lagrangian equations. Unfortunately, there is no general recipe of how to find such coordinates.

2.2.2 N-particle systems

Let's generalize things to be able to deal with systems with many particles and many degrees of freedom. We start by introducing a convenient nomenclature:

$$
\begin{array}{ccccccc}
\mathbf{r}_1 & & \mathbf{r}_2 & & \dots & & \mathbf{r}_N \\
x_1\ y_1\ z_1 & & x_2\ y_2\ z_2 & & & & x_N\ y_N\ z_N \\
\downarrow\ \downarrow\ \downarrow & & \downarrow\ \downarrow\ \downarrow & & & & \swarrow\ \downarrow\ \searrow \\
x_1\ x_2\ x_3 & & x_4\ x_5\ x_6 & & & & x_{3N-2}\ x_{3N-1}\ x_{3N} \\
m_1 & & m_2 & & & & m_N \\
\swarrow\ \downarrow\ \searrow & & \swarrow\ \downarrow\ \searrow & & & & \swarrow\ \downarrow\ \searrow \\
m_1 = m_2 = m_3 & & m_4 = m_5 = m_6 & & & & m_{3N-2} = m_{3N-1} = m_{3N}
\end{array}
$$

$$\text{yields, e.g. :} \qquad T = \sum_{i=1}^{N} \frac{m_i}{2} \mathbf{v}_i^2 = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{x}_i^2$$

a) Classification of constraints

(i) k_1 scleronomic-holonomic constraints

$$f_i(x_1, ..., , x_{3N}) = 0 \ ; \quad i = 1, ..., k_1$$

(ii) k_2 rheonomic-holonomic constraints

$$f_j(x_1, ..., , x_{3N}, t) = 0 \ ; \quad j = 1, ..., k_2$$

$\rightarrow$ $k_1 + k_2 = k$ holonomic constraints reduce the number of degrees of freedom of an N-particle system from $3N$ to $3N - k$ (if they are independent and compatible with each other).

(iii) m nonholonomic differential constraints
... can also be characterized by equations (see Appendix A.3), but they do not reduce the number of dofs.

b) Generalized coordinates and configuration space

goal: description of N-particle system in terms of suitable (generalized) coordinates

typical starting point: Cartesian coordinates

consider coordinate transformation $(x_1, ..., x_{3N}) \longleftrightarrow (q_1, ..., q_{3N})$

$x_i = x_i(q_1, ..., q_{3N}, t) \quad i = 1, ..., 3N$
generalized coordinates $q_\mu = q_\mu(x_1, ..., x_{3N}, t) \quad \mu = 1, ..., 3N$

assume k independent holonomic constraints
$f_j(x_1, ..., x_{3N}, t) = 0$ [or $f_j(q_1, ..., q_{3N}, t) = 0$], $j = 1, ..., k$

choose

$$\left.\begin{array}{ll} q_{3N-k+1} & = f_1(x_1, ..., x_{3N}, t) = 0 \\ q_{3N-k+2} & = f_2(x_1, ..., x_{3N}, t) = 0 \\ \vdots & \\ q_{3N} & = f_k(x_1, ..., x_{3N}, t) = 0 \end{array}\right\} \quad \text{(ignorable coordinates)}$$

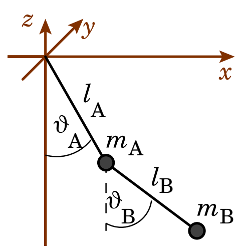
$\Longrightarrow 3N - k$ independent generalized coordinates remain. They describe the system completely.

$$\begin{array}{rrcl} & \boldsymbol{q} & = & (q_1, ..., q_{3N-k}): \quad \text{“configuration (vector)”} \\ & & = & \text{point in } (3N-k)\text{−dim. configuration space} \\ \hookrightarrow & x_i & = & x_i(q_1, ..., q_{3N-k}, t) \\ \hookrightarrow & \dot{x}_i & = & \dfrac{d}{dt} x_i(q_1, ..., q_{3N-k}, t) \\ & & = & \displaystyle\sum_{\mu=1}^{3N-k} \frac{\partial x_i}{\partial q_\mu}\, \dot{q}_\mu + \frac{\partial x_i}{\partial t} \\ & & = & \dot{x}_i(\mathbf{q}, \dot{\mathbf{q}}, t), \qquad i = 1, ..., 3N \end{array}$$

$\dot{q}_\mu$: generalized velocity

$$\hookrightarrow \quad L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

example: double pendulum ($N = 2$)



Cartesian coordinates $(x_1, ..., x_6)$

$$\left.\begin{array}{l} \text{constraints} : x_3 = x_6 \;=\; 0 \\ l_A = \sqrt{x_1^2 + x_2^2} \\ l_B = \sqrt{(x_4 - x_1)^2 + (x_5 - x_2)^2} \end{array}\right\} \Longrightarrow 4 \text{ constraints}$$

$$\Longrightarrow 2 \text{ dofs}$$

generalized coordinates:

$$\left.\begin{array}{l} q_1 = \vartheta_A \\ \\ q_2 = \vartheta_B \end{array}\right\} \text{characterize motion in } 2 - \text{dim. configuration space}$$

$$\left.\begin{array}{l} q_3 = x_3 = 0 \;,\;\; q_6 = x_6 = 0 \\ \\ q_4 = l_A - \sqrt{x_1^2 + x_2^2} = 0 \\ \\ q_5 = l_B - \sqrt{(x_4 - x_1)^2 + (x_5 - x_2)^2} = 0 \end{array}\right\} \text{ignorable}$$

$$L = T - U = \frac{1}{2} \sum_{i=1}^{6} m_i \dot{x}_i^2 - g(m_2 x_2 + m_5 x_5)$$

coordinate transformation:

$$\begin{array}{rrcl} & x_1 & = & l_A \sin\vartheta_A = l_A \sin q_1 \\ & x_2 & = & -l_A \cos\vartheta_A = -l_A \cos q_1 \\ & x_3 & = & 0 \\ & x_4 & = & l_A \sin q_1 + l_B \sin q_2 \\ & x_5 & = & -l_A \cos q_1 - l_B \cos q_2 \\ & x_6 & = & 0 \\ \text{general form} & x_i & = & x_i(q_1, q_2) \;, \quad i = 1, ..., 6 \end{array}$$

$$
\begin{aligned}
\dot{x}_1 &= l_A \dot{q}_1 \cos q_1 \\
\dot{x}_2 &= l_A \dot{q}_1 \sin q_1 \\
\dot{x}_3 &= 0 \\
\dot{x}_4 &= l_A \dot{q}_1 \cos q_1 + l_B \dot{q}_2 \cos q_2 \\
\dot{x}_5 &= l_A \dot{q}_1 \sin q_1 + l_B \dot{q}_2 \sin q_2 \\
\dot{x}_6 &= 0 \\
\text{general form} \qquad \dot{x}_i &= \dot{x}_i(q_1, q_2, \dot{q}_1, \dot{q}_2) \ , \quad i = 1, ..., 6
\end{aligned}
$$

use conventional nomenclature for the masses

$$
(m_1, m_2, m_3) \longrightarrow m_A \qquad (m_4, m_5, m_6) \longrightarrow m_B
$$

to obtain the Lagrangian

$$
\begin{aligned}
\hookrightarrow \quad L &= \frac{m_A}{2}\, l_A^2 \dot{q}_1^2 \Big(\cos^2 q_1 + \sin^2 q_1 \Big) + \frac{m_B}{2} \Big[\big(l_A \dot{q}_1 \cos q_1 + l_B \dot{q}_2 \cos q_2 \big)^2 \\
&\quad + \big(l_A \dot{q}_1 \sin q_1 + l_B \dot{q}_2 \sin q_2 \big)^2 \Big] + m_A g l_A \cos q_1 + m_B g \big(l_A \cos q_1 + l_B \cos q_2 \big) \\
&= \frac{m_A}{2}\, l_A^2 \dot{q}_1^2 + \frac{m_B}{2}\, l_A^2 \dot{q}_1^2 + \frac{m_B}{2}\, l_B^2 \dot{q}_2^2 + m_B l_A l_B \dot{q}_1 \dot{q}_2 \big(\cos q_1 \cos q_2 + \sin q_1 \sin q_2 \big) \\
&\quad + (m_A + m_B) g l_A \cos q_1 + m_B g l_B \cos q_2 \\
&= \frac{m_A + m_B}{2}\, l_A^2 \dot{q}_1^2 + \frac{m_B}{2}\, l_B^2 \dot{q}_2^2 + m_B l_A l_B \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2) \\
&\quad + (m_A + m_B) g l_A \cos q_1 + m_B g l_B \cos q_2 \\
&= L(q_1, q_2, \dot{q}_1, \dot{q}_2)
\end{aligned}
$$

2.3 General formulation of Hamilton's principle and Langrange's equations for N-particle systems

2.3.1 Hamilton's principle

Of all the possible paths along which a particle system with $3N$-k degrees of freedom may move from one point in configuration space to another one in a given time interval $[t_1, t_2]$ the actual path is such that the action integral

$$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\ dt$$

is stationary, i.e., $\delta S = 0$.

Comments:

(i) "Path": motion in $3N - k$ dimensional configuration space

$$\mathbf{q}(t) = \{q_1(t), ..., q_{3N-k}(t),\ t_1 \leq t \leq t_2\}$$

(ii) In most cases the stationary point is a minimum.

(iii) Derivation of Lagrangian equations of motion from HP
it is straightforward to generalize the arguments of Sec. 2.1.1

- assume that $\mathbf{q}(t)$ is the actual path for which $\delta S = 0$
- neighbouring paths:

$$\begin{array}{rrcll} & q_\mu(\alpha, t) & = & q_\mu(t) + \alpha\eta_\mu(t) & \qquad (\mu = 1, ..., 3N - k) \\ & \dot{q}_\mu(\alpha, t) & = & \dot{q}_\mu(t) + \alpha\dot{\eta}_\mu(t) & \\ \text{with} & \eta_\mu(t_1) & = & \eta_\mu(t_2) = 0 & \end{array}$$

- $\triangleleft \qquad S[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}(\alpha, t), \dot{\mathbf{q}}(\alpha, t), t)\ dt = S(\alpha)$
- $\mathbf{q}(t)$ is the sought-after path $\Leftrightarrow \left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0$

- $$\begin{aligned}\frac{dS}{d\alpha} &= \int_{t_1}^{t_2} \frac{\partial}{\partial\alpha}\, L(\mathbf{q}(\alpha,t),\dot{\mathbf{q}}(\alpha,t),t)\; dt \\ &= \sum_{\mu=1}^{3N-k} \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_\mu}\frac{\partial q_\mu}{\partial\alpha} + \frac{\partial L}{\partial \dot{q}_\mu}\frac{\partial \dot{q}_\mu}{\partial\alpha}\Big)\; dt \\ &= \sum_{\mu=1}^{3N-k} \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_\mu}\;\eta_\mu(t) + \frac{\partial L}{\partial \dot{q}_\mu}\;\dot{\eta}_\mu(t)\Big)\; dt \\ &= \underbrace{\sum_{\mu=1}^{3N-k} \frac{\partial L}{\partial \dot{q}_\mu}\;\eta_\mu(t)\Big|_{t_1}^{t_2}}_{=0} + \sum_{\mu=1}^{3N-k} \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu}\Big)\eta_\mu(t)\; dt\end{aligned}$$

$$\frac{dS}{d\alpha}\Big|_{\alpha=0} \Longleftrightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = 0\;, \quad \mu = 1,...,3N-k}$$

(for $q_\mu = q_\mu(\alpha = 0, t)$)

2.3.2 Equivalence of Lagrange's and Newton's equations of motion

a) System without constraints in Cartesian coordinates
show:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0\;, \quad i = 1,...,3N$$

$$\Longleftrightarrow \quad \dot{\mathbf{p}}_k = \mathbf{F}_k + \sum_{j=1}^{N} \mathbf{f}_{kj}\;, \quad k = 1,...,N$$

<u>proof</u>:

$$\begin{aligned}
L \;=\; T - U &= \frac{1}{2}\sum_{j=1}^{3N} m_j \dot{x}_j^2 - U(x_1, ..., x_{3N}) \\
\hookrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} &= \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} \;=\; m_i \ddot{x}_i \;=\; \dot{p}_i \quad (i = 1, ..., 3N) \\
\frac{\partial L}{\partial x_i} &= -\frac{\partial U}{\partial x_i} = \; F_i + \sum_{j=1}^{3N} f_{ij}
\end{aligned}$$

a detailed analysis of the last equation can be found in Appendix A.4.

b) Invariance of the Lagrangian equations to coordinate transformations

<u>goal</u>: show

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \quad \Longleftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = 0$$

with $\quad x_i = x_i(q_1, ..., q_{3N}, t)\ , \quad i = 1, ..., 3N$

$\Longrightarrow$ Lagrangian equations of motion in $3N$ generalized coordinates $\Longleftrightarrow$ Newtonian equations of motion in Cartesian coordinates

let's be a bit more general and show invariance of the Lagrangian equations with respect to general coordinate transformations

$$\begin{aligned}
q_\mu \to Q_\alpha &= Q_\alpha(\mathbf{q}, t)\ , \quad \alpha = 1, ..., n \\
q_\mu &= q_\mu(\mathbf{Q}, t)\ , \quad \mu = 1, ..., n
\end{aligned}$$

ingredient :
$$\begin{aligned}
\dot{q}_\mu &= \frac{d}{dt}\, q_\mu(Q_1, ..., Q_n, t) = \sum_{\beta=1}^{n} \frac{\partial q_\mu}{\partial Q_\beta}\ \dot{Q}_\beta + \frac{\partial q_\mu}{\partial t} \\
&= \dot{q}_\mu(Q_1 ... Q_n, \dot{Q}_1 ... \dot{Q}_n, t)
\end{aligned}$$

$$\hookrightarrow \qquad \frac{\partial \dot{q}_\mu}{\partial \dot{Q}_\alpha} = \frac{\partial}{\partial \dot{Q}_\alpha}\Big(\sum_\beta \frac{\partial q_\mu}{\partial Q_\beta}\dot{Q}_\beta + \frac{\partial q_\mu}{\partial t}\Big) = \frac{\partial q_\mu}{\partial Q_\alpha}$$

<u>assume</u> $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = 0\,, \quad \mu = 1, ..., n$$

$$\begin{aligned} L(q_1...q_n, \dot{q}_1...\dot{q}_n, t) &= L\Big(q_1(Q_1...Q_n, t), q_2(Q_1...Q_n, t), ..., \dot{q}_1(Q_1...Q_n, \dot{Q}_1...\dot{Q}_n, t), ..., t\Big) \\ &= \tilde{L}(Q_1...Q_n, \dot{Q}_1...\dot{Q}_n, t) \end{aligned}$$

<u>show</u> $$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_\alpha} - \frac{\partial \tilde{L}}{\partial Q_\alpha} = 0\,, \quad \alpha = 1, ..., n$$

- $$\frac{\partial \tilde{L}}{\partial Q_\alpha} = \sum_\mu \Big(\frac{\partial L}{\partial q_\mu}\frac{\partial q_\mu}{\partial Q_\alpha} + \frac{\partial L}{\partial \dot{q}_\mu}\frac{\partial \dot{q}_\mu}{\partial Q_\alpha}\Big)$$
- $$\frac{\partial \tilde{L}}{\partial \dot{Q}_\alpha} = \sum_\mu \frac{\partial L}{\partial \dot{q}_\mu}\frac{\partial \dot{q}_\mu}{\partial \dot{Q}_\alpha} = \sum_\mu \frac{\partial L}{\partial \dot{q}_\mu}\frac{\partial q_\mu}{\partial Q_\alpha}$$
- $$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_\alpha} = \frac{d}{dt}\Big(\sum_\mu \frac{\partial L}{\partial \dot{q}_\mu}\frac{\partial q_\mu}{\partial Q_\alpha}\Big) = \sum_\mu \left[\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_\mu}\Big)\frac{\partial q_\mu}{\partial Q_\alpha} + \frac{\partial L}{\partial \dot{q}_\mu}\underbrace{\frac{d}{dt}\frac{\partial q_\mu}{\partial Q_\alpha}}_{=\frac{\partial \dot{q}_\mu}{\partial Q_\alpha}}\right]$$

$$\longrightarrow \quad \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_\alpha} - \frac{\partial \tilde{L}}{\partial Q_\alpha} = \sum_\mu \left[\underbrace{\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_\mu}\Big) - \frac{\partial L}{\partial q_\mu}}_{=0}\right]\frac{\partial q_\mu}{\partial Q_\alpha} = 0 \qquad \text{(q.e.d.)}$$

<u>Comments</u>:

(i) Special case: $n = 3N$ und $Q_\alpha \equiv x_i$
$\hookrightarrow$ Lagrangian equations in $3N$ generalized coordinates $\Longleftrightarrow$ Newtonian equations in Cartesian coordinates

(ii) Newtonian equations of motion are invariant wrt Galilean transformations, but <u>not</u> wrt general coordinate transformations!

i.e. $\quad m_i \ddot{x}_i = F_i \quad$ does not imply $\quad m_\mu \ddot{q}_\mu = F_\mu$

(iii) Holonomic constraints

$$\text{assume} \qquad \mathbf{q} = \{q_1 ... q_{3N-k}, \underbrace{q_{3N-k-1} ... q_{3N}}_{\text{ignorable}}\}$$

$$\begin{aligned}
\text{HP}: \qquad \delta S &= 0 \quad \text{for} \quad L = L(q_1 ... q_{3N}, \dot{q}_1 ... \dot{q}_{3N}, t) \\
\Longleftrightarrow \qquad 0 &= \frac{dS}{d\alpha}\Big|_{\alpha=0} = \sum_{\mu=1}^{3N} \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu}\Big)\eta_\mu(t)\ dt \\
&= \sum_{\mu=1}^{3N-k} \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu}\Big)\eta_\mu(t)\ dt \\
&\quad + \sum_{\mu=3N-k+1}^{3N} \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu}\Big)\underbrace{\eta_\mu(t)}_{\substack{= 0 \text{ for } \mu = 3N-k+1,...,3N \\ \text{(ignorable coordinates are not varied)}}}\ dt
\end{aligned}$$

$$\Longleftrightarrow \qquad \boxed{\frac{\partial L}{\partial q_\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} = 0 \quad \text{for} \quad \mu = 1, ..., 3N-k}$$

$\longrightarrow$ HP $\Longleftrightarrow$ Lagrangian eqs. for $3N-k$ generalized coordinates $\Longleftrightarrow$ Newtonian eqs. for $3N$ Cartesian coordinates + forces of constraint (see [Tay], Chap. 7.4 for some more details on this)

(iv) Lagrangian recipe

- formulate k (holonomic) constraints
- choose $3N$ generalized coordinates, identify k of them with constraints
- set up $L = T - U$ in $3N$ Cartesian (or other) coordinates
- work out coordinate transformation to $3N-k$ generalized coordinates
- find $L = L(q_1 ... q_{3N-k}, \dot{q}_1 ... \dot{q}_{3N-k}, t)$

– work out

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = 0 \qquad (\mu = 1, ..., 3N - k)$$

– solve equations of motion and analyze solutions

2.4 Conservation theorems revisited

2.4.1 Generalized momenta

definition of a generalized (aka canonical, aka conjugate) momentum

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}$$

example 1: Cartesian coordinates

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2}\frac{\partial}{\partial x_i}\Big(\sum_j m_j \dot{x}_j^2\Big) = m_i \dot{x}_i$$

$\longrightarrow$ usual linear momentum component
example 2: pendulum in xy-plane

$$\begin{aligned} L &= \frac{m}{2} l^2 \dot{q}^2 + mgl\cos q \ , \quad (q = \varphi) \\ \hookrightarrow \quad p &= \frac{\partial L}{\partial \dot{q}} = ml^2 \dot{q} = ml^2 \dot{\varphi} = l_z = (\mathbf{r} \times \mathbf{p})_z \end{aligned}$$

$\longrightarrow$ z-component of the angular momentum

question: when is $\dot{p}_\mu = 0$?

$$\hookrightarrow \quad \text{Lagrangian eqs. :} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} = \dot{p}_\mu = \frac{\partial L}{\partial q_\mu}$$

$$\hookrightarrow \quad \boxed{\text{if} \quad \frac{\partial L}{\partial q_\mu} = 0 \quad \Rightarrow \quad \begin{cases} \dot{p}_\mu = 0 \\ \\ p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} = \text{const} \end{cases}}$$

example 1: free particle

$$\begin{aligned} L &= T = \frac{1}{2}\sum_i m_i \dot{x}_i^2 \\ &\hookrightarrow \frac{\partial L}{\partial x_i} = 0 \quad \Longleftrightarrow \quad p_i = m_i \dot{x}_i = \text{const} \end{aligned}$$

<u>example 2</u>: particle on the inside surface of a cone

let's assume that the cone (angle α) is standing upright with its symmetry axis parallel to the z-axis.

- constraint $\tan\alpha = \frac{\sqrt{x^2+y^2}}{z}$
- generalized (cylindrical) coordinates

$$\begin{aligned} q_1 &= r = \sqrt{x^2+y^2} \\ q_2 &= \varphi \\ q_3 &= \tan\alpha - \frac{r}{z} = 0 \quad \text{(ignorable)} \end{aligned}$$

- Lagrangian in Cartesian coordinates

$$L = T - U = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

- coordinate transformation

$$\begin{aligned} x &= r\cos\varphi = q_1 \cos q_2 \\ y &= r\sin\varphi = q_1 \sin q_2 \\ z &= r\cot\alpha = q_1 \cot\alpha \\ \dot{x} &= \dot{q}_1 \cos q_2 - q_1 \dot{q}_2 \sin q_2 \\ \dot{y} &= \dot{q}_1 \sin q_2 + q_1 \dot{q}_2 \cos q_2 \\ \dot{z} &= \dot{q}_1 \cot\alpha \end{aligned}$$

- Lagrangian in generalized coordinates

$$\begin{aligned} L &= \frac{m}{2}((\dot{q}_1 \cos q_2 - q_1\dot{q}_2 \sin q_2)^2 + (\dot{q}_1 \sin q_2 + q_1 \dot{q}_2 \cos q_2)^2 + \dot{q}_1^2 \cot^2\alpha) - mgq_1 \cot\alpha \\ &= \frac{m}{2}\left(\frac{\dot{q}_1^2}{\sin^2\alpha} + q_1^2\dot{q}_2^2\right) - mgq_1\cot\alpha \\ &= L(q_1, \dot{q}_1, \dot{q}_2) \end{aligned}$$

- conserved momentum

$$\frac{\partial L}{\partial q_2} = 0 \quad \hookrightarrow \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = mq_1^2\dot{q}_2 = mr^2\varphi = l_z = \text{const}$$

- remaining equation of motion

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} &= \frac{m\ddot{q}_1^2}{\sin^2\alpha}\\ \frac{\partial L}{\partial q_1} &= mq_1\dot{q}_2^2 - mg\cot\alpha\end{aligned}$$

$$\Longrightarrow \ddot{q}_1 - q_1\dot{q}_2^2\sin^2\alpha + g\sin\alpha\cos\alpha = 0$$

eliminate $\dot{q}_2$ by using $\dot{q}_2 = \frac{p_2}{mq_1^2}$

$$\hookrightarrow \ddot{q}_1 - \frac{p_2^2\sin^2\alpha}{m^2q_1^3} + g\sin\alpha\cos\alpha = 0$$

Comments:

(i) equation of motion in conventional notation

$$\ddot{r} - \frac{l_z^2\sin^2\alpha}{m^2r^3} + g\sin\alpha\cos\alpha = 0$$

(ii) alternate treatment with (z, φ) as generalized coordinates is also possible (and is very similar due to the linear relation between r and z)

(iii) a coordinate which is conjugate to a conserved momentum (such as $q_2 = \varphi$ in the previous example) is called a *cyclic* coordinate[1]

2.4.2 Energy and the Hamiltonian

a) Preparation 1: Euler's theorem for homogeneous functions

definition: $f(x_1...x_m)$ is a homogeneous function of degree n:

$$\Longleftrightarrow \quad f(\lambda x_1, \lambda x_2, ..., \lambda x_m) = \lambda^n f(x_1...x_m)$$

Euler: if f is a homogeneous function of degree n

$$\Longrightarrow \quad \sum_{i=1}^{m} x_i\frac{\partial f}{\partial x_i} = nf(x_1, ..., x_m)$$

[1]some authors call these coordinates ignorable

proof: $y_i = \lambda x_i$

$$\begin{aligned}
\sphericalangle \qquad \frac{\partial f}{\partial \lambda}\,(y_1, ..., y_m) &= \sum_i \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial \lambda} \\
&= \sum_i \frac{\partial f}{\partial y_i}\; x_i \\
&= \frac{\partial}{\partial \lambda}\,(\lambda^n f) \\
&= n\lambda^{n-1} f(x_1, ..., x_m)
\end{aligned}$$

for $\lambda = 1 \hookrightarrow y_i = x_i$ and

$$\sum_i x_i \frac{\partial f}{\partial x_i} = n f(x_1, ..., x_m) \ .$$

b) Preparation 2: a theorem concerning kinetic energy

theorem: For time-independent transformations $x_i \longleftrightarrow q_\mu$ $(x_i = x_i(\mathbf{q}))$ T is a homogeneous function of degree 2 in the generalized velocities.

$$\begin{aligned}
\text{proof :} \qquad T &= \frac{1}{2}\sum_i m_i \dot{x}_i^2 \\
&= \frac{1}{2}\sum_i m_i \sum_{\mu,\nu} \frac{\partial x_i}{\partial q_\mu}\frac{\partial x_i}{\partial q_\nu}\; \dot{q}_\mu\; \dot{q}_\nu \\
T(\lambda\dot{\mathbf{q}}) &= \lambda^2 T(\dot{\mathbf{q}})
\end{aligned}$$

$$\overset{\text{Euler}}{\Longrightarrow} \quad \boxed{\sum_\mu \dot{q}_\mu\; \frac{\partial T}{\partial \dot{q}_\mu} = 2T}$$

c) The Hamiltonian

The Lagrangian is not the total energy of a conservative system (unless $U = 0$) and normally not a conserved quantity. Still, it is useful to look at its total time derivative:

$$
\begin{aligned}
\frac{d}{dt}\, L(\mathbf{q},\dot{\mathbf{q}},t) \quad &= \quad \sum_\mu \left(\frac{\partial L}{\partial q_\mu}\,\dot q_\mu + \frac{\partial L}{\partial \dot q_\mu}\,\ddot q_\mu\right) + \frac{\partial L}{\partial t} \\
&\overset{\text{Lag.eqs.}}{=} \quad \sum_\mu \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\mu}\right)\,\dot q_\mu + \frac{\partial L}{\partial \dot q_\mu}\frac{d}{dt}\,\dot q_\mu\right] + \frac{\partial L}{\partial t} \\
&= \quad \frac{d}{dt}\left(\sum_\mu \frac{\partial L}{\partial \dot q_\mu}\,\dot q_\mu\right) + \frac{\partial L}{\partial t}
\end{aligned}
$$

$$
\Longleftrightarrow \quad \frac{d}{dt}\left\{\sum_\mu p_\mu \dot q_\mu - L\right\} = -\frac{\partial L}{\partial t}
$$

<u>definition</u>: Hamiltonian function

$$
\boxed{H = \sum_\mu p_\mu \dot q_\mu - L}
$$

$$
\hookrightarrow \quad \boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}
$$

$$
\Longrightarrow \quad \boxed{\text{if} \quad \frac{\partial L}{\partial t} = 0 \quad \Longrightarrow \quad \begin{cases} \dot H = 0 \\ H = \text{const} \end{cases}}
$$

<u>Discussion</u>:

(i) $\big[H\big] = \big[L\big] = \big[E\big] = \text{Joule}$

(ii) $H \equiv E = T + U$ if

- system conservative

- at most holonomic constraints
- time-independent transformation $x_i \;\to\; q_\mu$

the latter two conditions are normally realized through scleronomic constraints and resting reference frames

proof:

$$\begin{aligned} p_\mu = \frac{\partial L}{\partial \dot q_\mu} &= \frac{\partial T}{\partial \dot q_\mu} \quad (\frac{\partial U}{\partial \dot q_\mu} = 0) \\ \hookrightarrow \quad H = \sum_\mu p_\mu \dot q_\mu - L &= \sum_\mu \dot q_\mu \frac{\partial T}{\partial \dot q_\mu} - L = 2T - T + U \\ &= T + U \end{aligned}$$

(iii) conditions for $H = E = T + U$ are sufficient, but not necessary

(iv) $H = E$ and $\dot H = 0$ are independent
$\hookrightarrow$ it is possible that $\dot H = 0$ and $H \neq E$
$\hookrightarrow$ it is possible that $\dot H \neq 0$ and $H = E$

(iv) if conditions listed in (ii) are fulfilled and if $\frac{\partial L}{\partial t} = 0$

$$\Longrightarrow \quad \boxed{H = E = T + U = \text{const}}$$

d) Examples

(i) one-dimensional harmonic oscillator

$$\begin{aligned} &T = \frac{m}{2}\dot x^2 \ , \quad U = \frac{m}{2}\omega^2 x^2 \ , \quad L = T - U = \frac{m}{2}(\dot x^2 - \omega^2 x^2) \\ \hookrightarrow\ &H = p\dot x - L = m\dot x^2 - \frac{m}{2}\dot x^2 + \frac{m}{2}\omega^2 x^2 = \frac{m}{2}(\dot x^2 + \omega^2 x^2) = E = \text{const} \end{aligned}$$

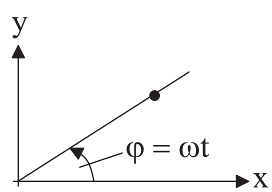
(ii) planar pendulum

$$\begin{aligned} L &= \frac{m}{2} l^2 \dot\varphi^2 + mgl\cos\varphi \\ \text{with} \quad p_\varphi &= ml^2 \dot\varphi \\ H &= p_\varphi \dot\varphi - L = ml^2\dot\varphi^2 - \frac{m}{2} l^2 \dot\varphi^2 - mgl\cos\varphi \\ &= \frac{m}{2} l^2 \dot\varphi^2 - mgl\cos\varphi = T + U = E = \text{const} \end{aligned}$$

(iii) planar double pendulum

$$
\begin{aligned}
T &= \frac{m_A+m_B}{2}\, l_A^2\dot\vartheta_A^2 + \frac{m_B}{2}\, l_B^2\dot\vartheta_B^2 + m_B l_A l_B \dot\vartheta_A\dot\vartheta_B\cos(\vartheta_A-\vartheta_B) \\
U &= -(m_A+m_B) g l_A\cos\vartheta_A - m_B g l_B\cos\vartheta_B \\
p_1 &= \frac{\partial L}{\partial\dot\vartheta_A} = (m_A+m_B) l_A^2\dot\vartheta_A + m_B l_A l_B\dot\vartheta_B\cos(\vartheta_A-\vartheta_B) \\
p_2 &= \frac{\partial L}{\partial\dot\vartheta_B} = m_B l_B^2\dot\vartheta_B + m_B l_A l_B\dot\vartheta_A\cos(\vartheta_A-\vartheta_B) \\
H &= p_1\dot\vartheta_A + p_2\dot\vartheta_B - T + U \\
&= (m_A+m_B) l_A^2\dot\vartheta_A^2 + m_B l_A l_B\dot\vartheta_A\dot\vartheta_B\cos(\vartheta_A-\vartheta_B) \\
&\quad + m_B l_B^2\dot\vartheta_B^2 + m_B l_A l_B\dot\vartheta_A\dot\vartheta_B\cos(\vartheta_A-\vartheta_B) \\
&\quad - T + U \;=\; T+U \;=\; E = \text{ const}
\end{aligned}
$$

(iv) bead on a rotating wire



rheonomic constraint : $y = x\tan\omega t \iff \varphi - \omega t = 0$

generalized coordinates $\begin{cases} q_1 = r \\ q_2 = \varphi - \omega t = 0 \quad \text{(ignorable)} \end{cases}$

- assume $U = 0$

$$
\begin{aligned}
L \;=\; T \;=\; \frac{m}{2}(\dot r^2 + r^2\dot\varphi^2) &= \frac{m}{2}(\dot r^2 + r^2\omega^2) = E \\
H \;=\; p\dot r - L = \frac{\partial L}{\partial\dot r}\,\dot r - L &= m\dot r^2 - \frac{m}{2}\dot r^2 - \frac{m}{2} r^2\omega^2 \\
&= \frac{m}{2}(\dot r^2 - r^2\omega^2) \neq E
\end{aligned}
$$

$$
-\frac{\partial L}{\partial t} = \frac{dH}{dt} = 0 \qquad \longrightarrow \qquad H = \text{const}
$$

Lagrangian equation of motion:

$$
\begin{aligned}
\underset{\displaystyle m\ddot r}{\underset{\|}{\frac{d}{dt}\frac{\partial L}{\partial\dot r}}} - \underset{\displaystyle m\omega^2 r}{\underset{\|}{\frac{\partial L}{\partial r}}} &= 0 \\
m\ddot r - m\omega^2 r &= 0 \iff \ddot r - \omega^2 r = 0
\end{aligned}
$$

general solution

$$\begin{aligned}
r(t) &= C_1 e^{\omega t} + C_2 e^{-\omega t} \\
\dot{r}(t) &= C_1 \omega e^{\omega t} - C_2 \omega e^{-\omega t} \\
\hookrightarrow \quad H &= \frac{m}{2}\Big\{\Big(C_1\omega e^{\omega t} - C_2\omega e^{-\omega t}\Big)^2 - \omega^2\Big(C_1 e^{\omega t} + C_2 e^{-\omega t}\Big)^2\Big\} \\
&= \frac{m}{2}\left\{C_1^2\omega^2 e^{2\omega t} + C_2^2\omega^2 e^{-2\omega t} - 2C_1C_2\omega^2 - C_1^2\omega^2 e^{2\omega t}\right. \\
&\quad \left. - C_2^2\omega^2 e^{-2\omega t} - 2\omega^2 C_1 C_2\right\} \\
&= -2m\omega^2 C_1 C_2 = \text{const}
\end{aligned}$$

$$\hookrightarrow \quad L = m\omega^2\Big(C_1^2 e^{2\omega t} + C_2^2 e^{-2\omega t}\Big) = E(t)$$

- assume $U = mgy = mgr\sin\omega t$

$$L = T - U = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t$$

$$\hookrightarrow \quad \tfrac{\partial L}{\partial t} = -mgr\omega\cos\omega t \neq 0 \quad \Longrightarrow \quad \dot{H} \neq 0$$

$$\begin{aligned}
H(t) &= \frac{m}{2}(\dot{r}^2 - r^2\omega^2) + mgr\sin\omega t \\
&\neq E(t) = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) + mgr\sin\omega t
\end{aligned}$$

L, H, E are all different and none is conserved!

(v) particle in a time-dependent homogeneous force field (1D)

$$\begin{aligned}
F(t) &= F_0 t \\
\hookrightarrow \quad U(x) &= -F_0 x t
\end{aligned}$$

$$\begin{aligned}
L &= T - U = \frac{m}{2}\dot{x}^2 + F_0 x t \\
H &= p\dot{x} - L = \frac{m}{2}\dot{x}^2 - F_0 x t = T + U = E = E(t)
\end{aligned}$$

$$\Big(\frac{\partial L}{\partial t} = F_0 x = -\dot{H}\Big)$$

2.5 Hamiltonian dynamics

We know that

$$L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

and

$$H = \sum_\mu p_\mu \dot{q}_\mu - L$$

Question: $H = H(?)$

To clarify on what variables H depends consider the total differential

$$\begin{aligned}
dH &= d\Big(\sum_\mu p_\mu \dot{q}_\mu - L(\mathbf{q}, \dot{\mathbf{q}}, t)\Big) \\
&= \sum_\mu \left\{ p_\mu d\dot{q}_\mu + \dot{q}_\mu dp_\mu - \frac{\partial L}{\partial q_\mu}\, dq_\mu - \frac{\partial L}{\partial \dot{q}_\mu}\, d\dot{q}_\mu \right\} - \frac{\partial L}{\partial t}\, dt \\
\text{use Lg.EoM :} \quad &= \sum_\mu \left\{ \dot{q}_\mu dp_\mu - \dot{p}_\mu dq_\mu \right\} - \frac{\partial L}{\partial t}\, dt \\
&= dH(\mathbf{q}, \mathbf{p}, t)
\end{aligned}$$

on the other hand:

$$dH(\mathbf{q}, \mathbf{p}, t) = \sum_\mu \left\{ \frac{\partial H}{\partial q_\mu}\, dq_\mu + \frac{\partial H}{\partial p_\mu}\, dp_\mu \right\} + \frac{\partial H}{\partial t}\, dt$$

$$\text{compare} \quad \Longrightarrow \quad \boxed{\dot{p}_\mu = -\frac{\partial H}{\partial q_\mu}\,, \qquad \dot{q}_\mu = \frac{\partial H}{\partial p_\mu}}$$

$\longrightarrow$ Hamilton's (canonical) equations of motion

Remarks:

(i) Hamilton's EoMs $\Longleftrightarrow$ Lagrange's EoMs $\Longleftrightarrow$ HP $\Longleftrightarrow$ Newton II show equivalence with Lagrange's EoMs:

$$
\begin{aligned}
L &= \sum_\mu p_\mu \dot{q}_\mu - H \\
\hookrightarrow \quad dL &= \sum_\mu \Big(p_\mu d\dot{q}_\mu + \dot{q}_\mu dp_\mu \Big) - dH(\mathbf{q}, \mathbf{p}, t) \\
&= \sum_\mu \Big(p_\mu d\dot{q}_\mu + \dot{q}_\mu dp_\mu - \frac{\partial H}{\partial q_\mu}\, dq_\mu - \frac{\partial H}{\partial p_\mu}\, dp_\mu \Big) - \frac{\partial H}{\partial t}\, dt \\
\text{use Hamilton's EoMs} \quad &= \sum_\mu \Big(p_\mu d\dot{q}_\mu + \dot{q}_\mu dp_\mu + \dot{p}_\mu dq_\mu - \dot{q}_\mu dp_\mu \Big) - \frac{\partial H}{\partial t}\, dt \\
&= dL(\mathbf{q}, \dot{\mathbf{q}}, t) \\
&= \sum_\mu \Big(\frac{\partial L}{\partial q_\mu} dq_\mu + \frac{\partial L}{\partial \dot{q}_\mu} d\dot{q}_\mu \Big) + \frac{\partial L}{\partial t}\, dt
\end{aligned}
$$

$$
\hookrightarrow \quad \left.\begin{array}{l} p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} \\[2ex] \dot{p}_\mu = \frac{\partial L}{\partial q_\mu} \end{array}\right\} \Longrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} = 0
$$

(ii) example 1: 1D oscillator

$$
\begin{aligned}
L &= T - U = \frac{m}{2}\Big(\dot{x}^2 - \omega^2 x^2 \Big) \\
H &= p\dot{x} - L = m\dot{x}\dot{x} - \frac{m}{2}\,\dot{x}^2 + \frac{m}{2}\,\omega^2 x^2 = \frac{m}{2}\Big(\dot{x}^2 + \omega^2 x^2 \Big) = T + U = E
\end{aligned}
$$

$$
p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Longleftrightarrow \quad \dot{x} = \frac{p}{m}
$$

$$
\hookrightarrow \quad H(x,p) = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2
$$

Note that $\dot{x}$ has been eliminated to make H a function of x and p. This is important to obtain the correct EoMs.

$$
\text{EoM}: \quad \left.\begin{array}{l} \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \\[2ex] \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \end{array}\right\} \longrightarrow \quad \begin{array}{c} \ddot{x} + \omega^2 x = 0 \\ \uparrow \\ \ddot{x} = \frac{\dot{p}}{m} \end{array}
$$

$$
\frac{\partial H}{\partial t} = 0 \quad \longrightarrow \quad H = E = \text{const}
$$

(iii) $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t} = \dfrac{dH}{dt} \quad \Longleftrightarrow \quad H = \text{const} \quad \text{if} \quad \dfrac{\partial H}{\partial t} = 0$

(iv) system with $n := 3N - k$ dofs:

$$\begin{aligned}
\text{Lagrange} \quad &\hat{=} \quad \text{n} - \text{ODEs of second order} \\
&\qquad (2n \text{ initial conditions } q_\mu(0), \dot{q}_\mu(0),\ \mu = 1, ..., n) \\
\text{Hamilton} \quad &\hat{=} \quad \text{2n} - \text{ODEs of first order} \\
&\qquad (2n \text{ initial conditions } q_\mu(0), p_\mu(0),\ \mu = 1, ..., n)
\end{aligned}$$

(v) the transformation from L to H is called a Legendre transformation

(vi) example 2: central-force problem in polar coordinates (to be discussed later)

$$L = T - U = \frac{m}{2}\big(\dot{r}^2 + r^2\dot{\varphi}^2\big) - U(r) = L\big(r, \dot{r}, \dot{\varphi}\big)$$

$$\begin{aligned}
\text{EoM :} \quad \frac{\partial L}{\partial \dot{r}} &= m\dot{r}\ , & \frac{\partial L}{\partial r} &= mr\dot{\varphi}^2 - \frac{\partial U}{\partial r} \\
&= p_r \\
\frac{\partial L}{\partial \dot{\varphi}} &= mr^2\dot{\varphi}\ , & \frac{\partial L}{\partial \varphi} &= 0 \qquad (\varphi \text{ cyclic}) \\
&= p_\varphi
\end{aligned}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} m\ddot{r} - mr\dot{\varphi}^2 + \frac{\partial U}{\partial r} &= 0 \\ r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} &= 0 \end{aligned}}$$

$$\begin{aligned}
H &= p_r\dot{r} + p_\varphi\dot{\varphi} - L(r, \dot{r}, \dot{\varphi}) \quad \text{with} \quad \dot{r} = \frac{p_r}{m} \quad \text{and} \quad \dot{\varphi} = \frac{p_\varphi}{mr^2} \\
&= p_r\,\frac{p_r}{m} + p_\varphi\,\frac{p_\varphi}{mr^2} - \frac{m}{2}\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\varphi}{mr^2}\right)^2\right) + U(r) \\
&= \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + U(r) \\
&= H(r, p_r, p_\varphi)
\end{aligned}$$

EoM:

$$\boxed{\begin{array}{rcl} \dot{p}_r &=& -\dfrac{\partial H}{\partial r} = \dfrac{p_\varphi^2}{mr^3} - \dfrac{\partial U}{\partial r}\,, \qquad \dot{p}_\varphi = -\dfrac{\partial H}{\partial \varphi} = 0 \\ \dot{r} &=& \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{m}\,, \qquad\qquad \dot{\varphi} = \dfrac{\partial H}{\partial p_\varphi} = \dfrac{p_\varphi}{mr^2} \end{array}}$$

$$\Longleftrightarrow \text{ Lagrange-EoM}$$

Note:

$$\left.\begin{array}{l} H = E = \frac{m}{2}\dot{r}^2 + \frac{p_\varphi^2}{2mr^2} + U(r) \\ \\ L = T - U = \frac{m}{2}\dot{r}^2 + \frac{p_\varphi^2}{2mr^2} - U(r) \end{array}\right\} \begin{array}{c} \text{correct but problematic :} \\ \text{only} \quad H = H(\mathbf{q},\mathbf{p},t) \quad \text{and} \quad L = L(\mathbf{q},\dot{\mathbf{q}},t) \\ \text{yield correct EoMs!} \end{array}$$

(vii) "phase space": populated by

$$\mathbf{\Pi} = (q_1 ... q_{3N-k}, p_1 ... p_{3N-k})$$

$\longrightarrow$ phase trajectorie $\mathbf{\Pi}(t)$ characterizes dynamics of the system

$$\mathbf{\Pi}(t_0) = \mathbf{\Pi}_0 \quad \xrightarrow{\text{Hamilton}-\text{EoMs}} \quad \mathbf{\Pi}(t)$$

$$\boxed{\text{state of a mechanical system} \quad \Longleftrightarrow \quad \text{phase } \mathbf{\Pi}}$$

$\longrightarrow$ all observables are functions of Π : $f = f(\mathbf{q},\mathbf{p},t) = f(\mathbf{\Pi},t)$

(viii) further reading: [Tay], Chap. 13

2.6 Extensions

2.6.1 Generalized forces and potentials

a) Formulation

$$\text{so far :} \qquad -\frac{\partial U}{\partial x_i} = F_i + \sum_{j=1}^{3N} f_{ij} \equiv \mathcal{F}_i \qquad i = 1, ..., 3N$$

let's translate this to generalized coordinates:

$$\begin{aligned} \lhd \qquad -\frac{\partial U}{\partial q_\mu} = -\sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\mu} &= \sum_i \mathcal{F}_i \frac{\partial x_i}{\partial q_\mu} \\ &\equiv Q_\mu \qquad \text{“generalized force components''} \\ \Big(&= Q_\mu(q_1 ... q_{3N-k}, t)\Big) \end{aligned}$$

standard Lagrangian: $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}, t)$

$$\text{EoMs :} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\mu} = \frac{\partial L}{\partial q_\mu} = \frac{\partial T}{\partial q_\mu} - \frac{\partial U}{\partial q_\mu}$$

$$\Longleftrightarrow \qquad \boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu} = Q_\mu \ , \quad \mu = 1, ..., 3N - k}$$

$\longrightarrow$ alternate form of the Lagrangian equations of motion

extension: consider a *generalized* (velocity-dependent) potential $U^* = U^*(\mathbf{q}, \dot{\mathbf{q}}, t)$

$$\text{with} \quad Q_\mu = -\left(\frac{\partial U^*}{\partial q_\mu} - \frac{d}{dt}\frac{\partial U^*}{\partial \dot{q}_\mu}\right)$$

$$\longrightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu} = \frac{d}{dt}\frac{\partial U^*}{\partial \dot{q}_\mu} - \frac{\partial U^*}{\partial q_\mu}$$

$$\Longleftrightarrow \qquad \boxed{\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} &= 0 \\ \text{for} \qquad L &= T - U^* \end{aligned}}$$

b) Application: charged particle in an electromagnetic (EM) field

- the Lorentzian force

$$\mathbf{F}_L = q\big(\mathbf{E} + (\mathbf{v} \times \mathbf{B})\big)$$

is obviously velocity dependent. One can construct a generalized potential U^* such that $F_L^i = -\Big(\frac{\partial U^*}{\partial x_i} - \frac{d}{dt}\frac{\partial U^*}{\partial \dot{x}_i}\Big)$. To do so we need to move from the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ to the corresponding potentials:

- EM potentials

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned}$$

$\phi, \mathbf{A}$: scalar and vector potentials

$$\hookrightarrow \mathbf{F}_L = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \times (\nabla \times \mathbf{A})))$$

- generalized potential

$$\begin{aligned} U^* &= q\big(\phi - \mathbf{v} \cdot \mathbf{A}\big) \\ \hookrightarrow F_L^i &= -\Big(\frac{\partial U^*}{\partial x_i} - \frac{d}{dt}\frac{\partial U^*}{\partial \dot{x}_i}\Big) \quad (i = 1, 2, 3) \end{aligned}$$

- Lagrangian

$$\begin{aligned} L &= T - U^* \\ &= \frac{m}{2}\mathbf{v}^2 - q\phi + q\mathbf{v} \cdot \mathbf{A} \end{aligned}$$

one can show (try it!) that the Lagrangian equations of motion are nothing else but Newton's equations for the Lorentzian force. This shows the consistency of the argument.

- Hamiltonian

$$H = \mathbf{p} \cdot \mathbf{v} - L$$

with the generalized momentum components

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i$$

note that the generalized momentum vector is in general not parallel to the velocity vector (due to the occurrence of the vector potential)

$$\begin{aligned}
\hookrightarrow H &= (m\mathbf{v} + q\mathbf{A}) \cdot \mathbf{v} - \frac{m}{2}\mathbf{v}^2 + q\phi - q\mathbf{v} \cdot \mathbf{A} \\
&= \frac{m}{2}\mathbf{v}^2 + q\phi \\
&= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi \\
&= E
\end{aligned}$$

obviously, H is the total energy of the system (kinetic energy plus 'electric' potential energy; the magnetic field (vector potential) does not exert work on the charged particle. However, in general (time-dependent fields) the energy is not conserved.

further reading: [GPS], Chaps. 1.5, 8.1

2.6.2 Friction

Fricitional forces are also velocity dependent, but they cannot be associated with a generalized potential. However, one can take them into account by introducing another function into the Lagrangian equations.

$$\text{assume :} \quad F_i^{\text{fric}} = -\beta_i \dot{x}_i$$

$$\begin{aligned}
\hookrightarrow \quad Q_\mu^{\text{fric}} &= \sum_i F_i^{\text{fric}} \frac{\partial x_i}{\partial q_\mu} = -\sum_i \beta_i \dot{x}_i \frac{\partial x_i}{\partial q_\mu} \\
&= -\sum_i \beta_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_\mu} = -\frac{\partial}{\partial \dot{q}_\mu} \Big(\sum_i \frac{\beta_i}{2} \dot{x}_i^2 \Big)
\end{aligned}$$

<u>definition</u>: Rayleigh's dissipation function

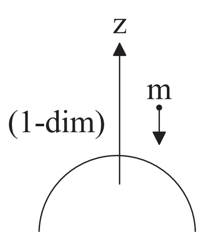
$$R := \sum_i \frac{\beta_i}{2} \dot{x}_i^2 = R(\mathbf{q}, \dot{\mathbf{q}}, t)$$

$$\text{Lag. eqs. :} \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu} = Q_\mu = Q_\mu^{\text{con}} + Q_\mu^{\text{fric}}$$
$$= -\frac{\partial U}{\partial q_\mu} - \frac{\partial R}{\partial \dot{q}_\mu}$$

$$\Longleftrightarrow \quad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} + \frac{\partial R}{\partial \dot{q}_\mu} = 0}$$

note the difference to the previous case of generalized potentials U^*: in the present case we have a standard Lagrangian $L = T - U$ and the frictional force enters the equations of motion thorugh an additional term. In the former case the velocity-dependent force is accounted for by a non-standard potential, but the form of the Lagrangian equations of motion is unchanged.

example: free fall with air resistance (1D)



$$L = \frac{m}{2}\dot{z}^2 - mgz$$
$$R = \frac{\beta}{2}\dot{z}^2$$

$$\Longrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z} \ , \quad \frac{\partial L}{\partial z} = -mg \ , \quad \frac{\partial R}{\partial \dot{z}} = \beta\dot{z}$$
$$\stackrel{\text{Lag. eq.}}{\Longrightarrow} \quad m\ddot{z} + mg + \beta\dot{z} = 0$$

for the solution of this EoM see, e.g. [FC], Chap. 4.3
for (a bit) more on the dissipation function see [GPS], Chap. 1.5

Let's end this chapter with mentioning two further topics, which are of some interest, but which we do not cover (for time reasons).

2.6.3 Lagrange's equations with undetermined multipliers

This variant can be used if the constraints are given in differential form. It enables the calculation of the forces of constraint, which is of interest from

an applied (engineering) point of view.
further reading: [TM], Chap. 7.5; [Tay], Chap. 7.10

2.6.4 d'Alembert's principle

This is an independent postulate (involving strange things like virtual displacements and virtual work), which can be used to derive the Lagrangian equations of motion without invoking Hamilton's principle.
further reading: [GPS], Chap. 1.4

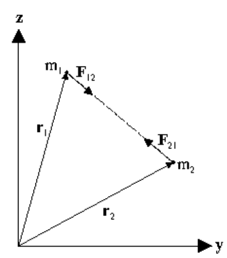
Chapter 3

Applications

3.1 Central-force problem

3.1.1 Preliminary

Consider the following situation:



1. isolated system ($\mathbf{F}_{ext} = 0$)
2. $\mathbf{f}_{12} = -\mathbf{f}_{21}$
   $= -\nabla_1 \bar{U}(|\mathbf{r}_1 - \mathbf{r}_2|) = \nabla_2 \bar{U}(|\mathbf{r}_1 - \mathbf{r}_2|)$
3. no constraints $\rightarrow$ 6 dofs

$$\begin{aligned}
\hookrightarrow \quad \text{Lagrangian :} \quad L &= T - \bar{U} = \frac{m_1}{2}\mathbf{v}_1^2 + \frac{m_2}{2}\mathbf{v}_2^2 - \bar{U}(|\mathbf{r}_1 - \mathbf{r}_2|) \\
\text{gravity :} \quad \bar{U}(|\mathbf{r}_1 - \mathbf{r}_2|) &= -G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \\
G &= 6.67 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}
\end{aligned}$$

solar system: Sun + eight planets (plus smaller objects)
let's estimate the mutual forces. First, a few masses (in units of the mass of the Earth):

Sun	$M_\odot$	=	$330.000\, m_E$
mercury (the smallest planet)	M_{Me}	=	$\frac{1}{20}\, m_E$
jupiter (the biggest planet)	M_{Ju}	=	$320\, m_E$

$$\triangleleft \quad \frac{\text{force on Earth due to Sun}}{\text{force on Earth due to X}} = \frac{f_{E\odot}}{f_{EX}} = \frac{M_\odot}{m_X}\frac{R_{XE}^2}{R_{\odot E}^2}$$

	Venus	Mars	Jupiter	Moon
m_X $[m_E]$	0.81	0,11	320	0.012
R_{XE}^{min} $[R_{\odot E}]$	0.27	0.52	4.2	0.0026
$f_{E\odot}/f_{EX}$	30000	810000	18300	180

$\Longrightarrow$ it is a good (first-order) approximation to view the Earth-Sun problem as an isolated two-body problem

3.1.2 Reduction of the two-body problem to an effective one-body problem

$$\begin{aligned} \text{momentum conservation :} \quad \dot{\mathbf{P}} &= \mathbf{F}_{ext} = 0 \\ \text{with} \quad \mathbf{P} &= M\mathbf{V} = (m_1+m_2)\dot{\mathbf{R}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \\ \mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r_2}}{m_1+m_2} \quad \text{centre of mass (CM)} \end{aligned}$$

The CM moves uniformly. Three coordinates are necessary to describe this motion. Three coordinates are then left to describe the internal dynamics of the system — the relative motion of the two bodies.
Positions wrt. CM: $\mathbf{r}'_k = \mathbf{r}_k - \mathbf{R} \quad (k = 1, 2)$

$$\begin{aligned} \hookrightarrow \quad T &= \frac{1}{2}\sum_k m_k\mathbf{v}_k^2 = \frac{1}{2}\sum_k m_k(\mathbf{v}'_k + \mathbf{V})^2 \\ &= \frac{1}{2}\sum_k m_k\mathbf{V}^2 + \frac{1}{2}\sum_k m_k\mathbf{v}'^2_k + \underbrace{\sum_k m_k\mathbf{v}'_k \cdot \mathbf{V}}_{=0} \end{aligned}$$

$$\hookrightarrow \quad T = T_{CM} + T' \; ; \quad T_{CM} = \frac{1}{2}M\mathbf{V}^2 \; ; \quad T' = \frac{1}{2}\sum_k m_k\mathbf{v}'^2_k \quad (\text{holds for } N \geq 2)$$

For $N = 2$ one can further rewrite the expressions:

$$
\begin{aligned}
\sphericalangle \quad \mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R} = \mathbf{r}_1 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{(m_1+m_2)\mathbf{r}_1 - m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1+m_2} \\
&= \frac{m_2}{m_1+m_2}\,(\mathbf{r}_1 - \mathbf{r}_2) = \frac{m_2}{m_1+m_2}\,\mathbf{r} \\
\mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{R} = \mathbf{r}_2 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{(m_1+m_2)\mathbf{r}_2 - m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1+m_2} \\
&= -\frac{m_1}{m_1+m_2}\,\mathbf{r}
\end{aligned}
$$

$$\text{with} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2' \qquad \text{“relative vector''}$$

$$\hookrightarrow \quad \mathbf{v}_1' = \frac{m_2}{m_1+m_2}\mathbf{v} \ ; \quad \mathbf{v}_2' = -\frac{m_1}{m_1+m_2}\mathbf{v}$$

$$
\begin{aligned}
\hookrightarrow \quad T' &= \frac{m_1}{2}\mathbf{v}_1'^2 + \frac{m_2}{2}\mathbf{v}_2'^2 = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\mathbf{v}^2 = \frac{1}{2}\mu\mathbf{v}^2 \\
& \quad \mu = \frac{m_1 m_2}{m_1+m_2} \qquad \text{“reduced mass''} \\
\rightarrow \quad T &= \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2 \ ; \quad \bar{U} = -G\frac{\mu M}{r}
\end{aligned}
$$

$$
\begin{aligned}
\longrightarrow \quad L &= \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{\mu}{2}\dot{\mathbf{r}}^2 + G\frac{\mu M}{r} \\
&= L\left(\dot{\mathbf{R}}, \mathbf{r}, \dot{\mathbf{r}}\right) \\
&= L_{CM}(\dot{\mathbf{R}}) + L_{rel}(\mathbf{r}, \dot{\mathbf{r}})
\end{aligned}
$$

The first three Lagrangian equations are simple (and contain no news):

$$
\begin{array}{lll}
\frac{\partial L}{\partial X} = \frac{\partial L}{\partial Y} = \frac{\partial L}{\partial Z} = 0 & \Longrightarrow & \left.\begin{array}{l} P_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} = \text{const} \\ P_Y = \frac{\partial L}{\partial \dot{Y}} = M\dot{Y} = \text{const} \\ P_Z = \frac{\partial L}{\partial \dot{Z}} = M\dot{Z} = \text{const} \end{array}\right\} \begin{array}{l}\text{total} \\ \text{momentum} \\ \text{conservation}\end{array} \\
(X, Y, Z \quad \text{cyclic}) & &
\end{array}
$$

3.1.3 Relative motion

a) Lagrangian

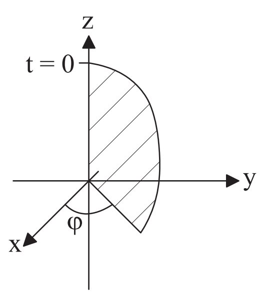
choose spherical coordinates (r, θ, φ)

$$L_{rel}(r,\theta,\dot r,\dot\theta,\dot\varphi) = \frac{\mu}{2}\left[\dot r^2 + (r\dot\varphi\sin\theta)^2 + (r\dot\theta)^2\right] + G\frac{\mu M}{r}$$

generalized momenta:

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot r} = \mu \dot r \\ p_\theta &= \frac{\partial L}{\partial \dot\theta} = \mu r^2\dot\theta \\ p_\varphi &= \frac{\partial L}{\partial \dot\varphi} = \mu r^2(\sin^2\theta)\dot\varphi = \text{const} \qquad (\varphi \text{ cyclic}) \end{aligned}$$

choose a reference system such that $x(0) = y(0) = 0 \quad (\rightarrow \theta(0) = 0)$



$$p_\varphi(0) = 0 = p_\varphi(t) \quad \Longrightarrow \quad \dot\varphi = 0$$

$$\Longrightarrow \quad \text{2D motion in plane } \varphi = \text{const}$$

$$\hookrightarrow \quad \boxed{L_{rel} = \frac{\mu}{2}(\dot r^2 + r^2\dot\theta^2) + G\frac{\mu M}{r}}$$

now θ is also cyclic and $p_\theta = \mu r^2\dot\theta \; = \text{const}$

Newtonian view on angular momentum conservation:

$$\mathbf{l} = \mu(\mathbf{r}\times\mathbf{v}) = \text{const} \quad \text{for central forces}$$

the conservation of the angular momentum *vector* implies

(i) conservation of direction $\rightarrow$ motion in a plane ($\perp \mathbf{l}$)

(ii) conservation of magnitude

$$l = \mu|\mathbf{r}\times\mathbf{v}| = \mu r^2\dot\theta \; = p_\theta$$

this also has a geometrical interpretation: the area per unit time swept by $\mathbf{r}(t)$ is given by

$$\dot{A} = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| = \frac{l}{2\mu} = \text{const}$$

this is nothing else but Kepler's second law (see below).

Let's work out the equation of motion:

$$\frac{\partial L_{rel}}{\partial \dot{r}} = \mu \dot{r} \ , \quad \frac{\partial L_{rel}}{\partial r} = \mu r \dot{\theta}^2 - G\frac{\mu M}{r}$$

$$\begin{aligned} \stackrel{\text{EoM}}{\Longrightarrow} \quad \mu \ddot{r} &= \mu r \dot{\theta}^2 - G\frac{\mu M}{r^2} \\ \Longleftrightarrow \quad \mu \ddot{r} &= \frac{l^2}{\mu r^3} - G\frac{\mu M}{r^2} \end{aligned} \tag{3.1}$$

b) Hamiltonian, energy, and a qualitative discussion of the Kepler orbits

$$\begin{aligned} H_{rel} &= p_r \dot{r} + l\dot{\theta} - L_{rel} \\ &= \ .... \ = \frac{\mu}{2}(\dot{r}^2 + \dot{r}^2\dot{\theta}^2) - G\frac{\mu M}{r} \\ &= \frac{p_r^2}{2\mu} + \frac{l^2}{2\mu r^2} - G\frac{\mu M}{r} \ = \ E_{rel} \ = \ \text{const} \quad \text{(energy conservation)} \\ & \quad \left(\text{since } -\frac{\partial \mathrm{L}}{\partial \mathrm{t}} = \dot{\mathrm{H}} = 0\right) \\ &= T_{rad} + T_{rot} + U_{grav} \\ &= T_{rad} + U_{eff}(r) \\ \text{with } U_{eff}(r) &= \frac{l^2}{2\mu r^2} - G\frac{\mu M}{r} \end{aligned}$$

$$T_{rad} = E_{rel} - U_{eff}(r) \geq 0$$

$$\Longleftrightarrow \quad \boxed{E_{rel} \geq U_{eff}(r)}$$

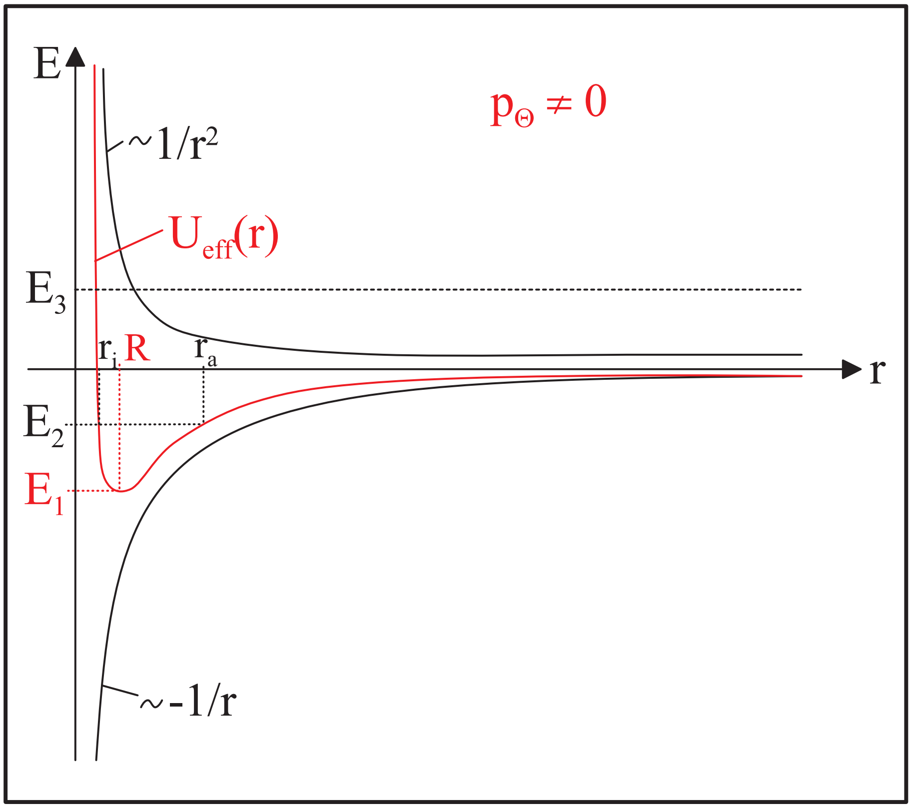


Figure 3.1: Effective potential of the Kepler problem (schematic)

From now on use $E \equiv E_{rel}$ (and $l \equiv p_\theta$)

(i) $E = E_1 = U_{eff}(R) \quad \longrightarrow \quad T_{rad} = 0$

$\longrightarrow$ circular orbit with angular velocity $\dot{\theta} = \frac{l}{\mu R^2} = \text{const}$

(ii) $E_1 < E = E_2 < 0$

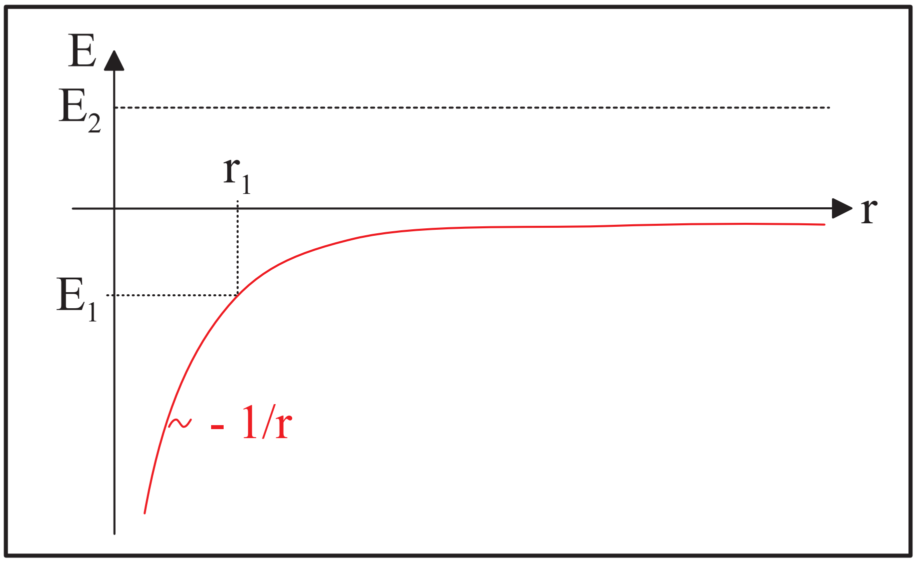
$\longrightarrow$ finite (bounded) orbit in $[r_i, r_a]$

(r_i, r_a : turning points of radial motion, $T_{rad}(r_i) = T_{rad}(r_a) = 0$) with changing angular velocity $\dot{\theta} = \frac{l}{\mu r^2}$

(iii) $E = E_3 \geq 0$

infinite (unbounded) orbit in $[r_3, \infty)$, (in general $r \stackrel{t\to\infty}{\longrightarrow} \infty$)

$\triangleleft \qquad l = 0:$



(i) $E = E_1 < 0$ finite (bounded) orbit in $[0, r_1]$ (cf. free fall)

(ii) $E = E_2 > 0$ infinite (unbounded) orbit (in general $r \stackrel{t\to\infty}{\longrightarrow} \infty$)

Remarks:

(i) Analysis refers to 'quasiparticle' of mass μ and relative motion. For the Earth-Sun system we have

$$\mu = \frac{M_\odot m_E}{M_\odot + m_E} \approx m_E \ ; \quad \mathbf{r} \approx \mathbf{r}_E \ ; \quad M \approx M_\odot \ ; \quad \mathbf{R} \approx \mathbf{r}_\odot$$

and an orbit of type (ii) (for the case $l \neq 0$)

(ii) Similar analysis is possible for other (central) potentials and is also useful for quantum-mechanical problems

(iii) Quantitative analysis requires solution of EoM

(iv) Further reading: e.g. [TM], Chap. 8.5, 8.6

c) Quantitative analysis

$$\text{starting point :} \qquad E = \frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

$$\begin{aligned}
\Longleftrightarrow \qquad \dot{r} &= \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\Big(E - U(r) - \frac{l^2}{2\mu r^2}\Big)} \\
\longrightarrow \quad t - t_0 &= \int_{t_0}^{t} dt' \;=\; \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{\mu}\Big(E - U(r') - \frac{l^2}{2\mu r'^2}\Big)}}
\end{aligned}$$

$\longrightarrow$ inversion yields $r(t)$

$$\text{obtain } \theta(t) \text{ from} \quad \dot{\theta} = \frac{l}{\mu r^2} \qquad \longrightarrow \qquad \theta(t) - \theta_0 = \frac{l}{\mu}\int_{t_0}^{t} \frac{dt'}{r^2(t')}$$

the problem with this procedure is that the integral cannot be calculated in closed analytical form (for the gravitational potential).

alternative consideration: analyze orbits $r(\theta)$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{l}{\mu r^2}\,\frac{dr}{d\theta} = \pm\sqrt{\frac{2}{\mu}\Big(E - U(r) - \frac{l^2}{2\mu r^2}\Big)}$$

$$\hookrightarrow \quad \theta(r) - \theta_0 = \int_{\theta_0}^{\theta} d\theta' = \pm\frac{l}{\mu}\int_{r_0}^{r} \frac{dr'}{r'^2}\,\frac{1}{\sqrt{\frac{2}{\mu}\Big(E - U(r') - \frac{l^2}{2\mu r'^2}\Big)}}$$

inversion $\longrightarrow$ $r(\theta)$

for $U(r) = -\frac{k}{r}$ $(k = G\mu M)$ this integral can be solved by using the substitution $u = 1/r$ and the indefinite integral

$$\int \frac{du}{\sqrt{a + bu + cu^2}} = \frac{1}{\sqrt{-c}}\arccos\left(-\frac{b + 2cu}{\sqrt{b^2 - 4ac}}\right)$$

One obtains the *Kepler orbits*: conic sections (in polar coordinates) with one focus at the origin (look up 'conic sections' on `wikipedia`!)

$$\boxed{\begin{aligned} \frac{1}{r} &= \frac{1}{\alpha}\Big(1+\varepsilon\cos(\theta-\theta')\Big) \\ \alpha &= \frac{l^2}{\mu k} > 0 \\ \varepsilon &= \sqrt{1+\frac{2El^2}{\mu k^2}} \geq 0 \qquad \text{“eccentricity”} \end{aligned}}$$

Classification of Kepler orbits

$$\left.\begin{array}{l|l|l} \varepsilon = 0 & E = -\frac{\mu k^2}{2l^2} \equiv E_{circ} & \text{circle} \\ & r = \alpha = \frac{l^2}{\mu k} = \text{const} & \\ \hline & E_{circ} < E < 0 & \\ 0 < \varepsilon < 1 & r_{max} = \frac{\alpha}{1-\varepsilon}\ , \quad r_{min} = \frac{\alpha}{1+\varepsilon} & \text{ellipse} \\ & \quad \text{“aphelion”} \qquad\qquad \text{“perihelion”} & \end{array}\right\} \begin{array}{c} \text{planets} \\ (+\text{ comets}) \end{array}$$

$$\left.\begin{array}{c|c|l} \varepsilon = 1 & E = 0 & \text{parabola} \\ \varepsilon > 1 & E > 0 & \text{hyperbola} \end{array}\right\} \text{comets}$$

d) Kepler's laws (1609, 1619)

I. “Planets move in elliptical orbits about the Sun with the Sun at one focus.”

II. “The area per unit time swept out by the radius vector from the Sun to a planet is constant.”

III. “The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.”

e) Further remarks and references

(i) Instead of using energy conservation (the first integral of the motion) one can obtain the Kepler orbits directly from the EoM: Let's first rewrite $\ddot{r}$:

$$\begin{aligned}
\dot{r} &= \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{l}{\mu r^2}\,\frac{dr}{d\theta} \\
\hookrightarrow\ \ddot{r} &= \frac{d}{dt}\left(\frac{l}{\mu r^2}\,\frac{dr}{d\theta}\right) = \dot{\theta}\frac{d}{d\theta}\left(\frac{l}{\mu r^2}\frac{dr}{d\theta}\right) = \frac{l^2}{\mu^2 r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) \\
&\overset{r=1/u}{=} \frac{l^2}{\mu^2}u^2\frac{d}{d\theta}\left(u^2\frac{dr}{du}\frac{du}{d\theta}\right) = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}.
\end{aligned}$$

Plug this result into the EoM (3.1) to obtain

$$\frac{d^2u(\theta)}{d\theta^2} + u(\theta) = \frac{\mu k}{l^2} = \alpha^{-1}.$$

This is a relatively simple inhomogeneous differential equation that can be solved without great difficulty (try it!). It yields — of course — the same results for the Kepler orbits as spelled out on the previous page ([Tay], Chap. 8.6).

(ii) An analysis of the motion as a function of time can be found in [GPS], Chap. 3.8.

(iii) It turns out that the Kepler problem has another constant of motion: the *Laplace-Runge-Lenz* (LRL) vector $\mathbf{A}$:

$$\mathbf{A} = \mathbf{p}\times\mathbf{l} - \mu k\frac{\mathbf{r}}{r}$$

where $\mathbf{p}$ and $\mathbf{l}$ are the linear and the angular momentum vectors and $k = G\mu M$.

<u>proof</u> :

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} &\overset{\dot{\mathbf{l}}=0}{=} \dot{\mathbf{p}}\times\mathbf{l} - \mu k\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \\
&= -\frac{\mu k}{r^3}(\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}})) - \mu k\left(\frac{r\dot{\mathbf{r}} - \dot{r}\mathbf{r}}{r^2}\right) \\
&= -\mu k\frac{\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}}) - r^2\dot{\mathbf{r}} + r^2\dot{\mathbf{r}} - \dot{r}r\mathbf{r}}{r^3} = 0
\end{aligned}$$

Note that in the second step Newton's EoM in the form $\dot{\mathbf{p}} = -\frac{k}{r^2}\frac{\mathbf{r}}{r}$ was used. It looks now as if we had too many conserved quantities: $\mathbf{l}, \mathbf{A}, E$ involve seven components, which cannot all be independent. It turns out that only five of them are; see [GPS], Chap. 3.9 (this chapter also includes a discussion of further properties of the LRL vector).

(iv) Relative motion $\rightarrow$ two-body motions
Keep in mind that the Kepler orbits are associated with the relative motion of the two-body problem. If $m_1 \gg m_2$ the relative motion is a good approximation of the motion of m_2, while m_1 is always close to the centre of mass (CM). If the two masses are not that different one has to consider the coordinate transformations discussed in Sec. 3.1.2. If the orbits are bounded, one obtains similar ellipses of both bodies about the CM.

(v) Deviations from the ideal elliptical orbits in the solar system are observable. They are (with decreasing importance) due to

- gravitational forces among the planets
- effects associated with Einstein's theory of general relativity
- solar oblateness (deviations of the mass distribution of the sun from a sphere)

(vi) Further reading: [TM], Chap. 8; [Tay], Chap. 8 (both book chapters contain proofs of Kepler's third law).

3.2 Dynamics of rigid bodies

3.2.1 Preparations

definition: "rigid body"
A rigid body is an aggregate of particles (mass points), whose relative distances are constrained to remain (absolutely) fixed.

$\lhd$ rigid body consisting of N mass points at positions $(\mathbf{r}_1 ... \mathbf{r}_N)$:

constraints:

$$|\mathbf{r}_i - \mathbf{r}_j| = c_{ij} = \text{const} \quad \forall\, ij \longrightarrow \begin{pmatrix} \text{N} \\ 2 \end{pmatrix} = \tfrac{N(N-1)}{2} \quad \text{constraints}$$

N	$\begin{pmatrix} \text{N} \\ 2 \end{pmatrix}$	$3N - \begin{pmatrix} \text{N} \\ 2 \end{pmatrix}$
2	1	5
3	3	6
4	6	6
5	10	5
6	15	3
7	21	0
8	28	-4

$\longrightarrow \quad \begin{pmatrix} N \\ 2 \end{pmatrix}$ constraints cannot be independent!

In fact, one can show that for $N > 2$ one has (in general)

$$3N - 6 \text{ independent constraints} \qquad \Longrightarrow \qquad \boxed{6 \quad \text{dofs}}$$

Lagrangian:

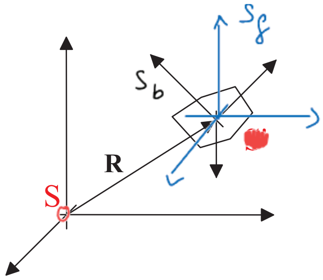
$$L = T - U \;=\; \frac{1}{2}\sum_i^N m_i \mathbf{v}_i^2 - \sum_{i=1}^N U(\mathbf{r}_i)$$

Note that internal forces—if present—are neutralized by the constraints and do not contribute.

<u>Chasles' theorem:</u> the general motion of a rigid body is composed of a translation of a point and a rotation about that point.

The natural choice for the 3 coordinates of the translational motion are the (Cartesian) coordinates of the CM: $\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$

Let's introduce three reference frames:



S_o: fixed (inertial) system

S_f: non-rotating system with fixed axes and with CM as origin

S_b: rotating body system also with CM as origin

coordinates and velocity of i-th mass point:

$$\begin{aligned} \mathbf{r}_i\Big|_{S_o} &= \mathbf{R} + \mathbf{r}_i\Big|_{S_{f,b}} \\ \mathbf{v}_i\Big|_{S_o} &= \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}_i\Big|_{S_{f,b}} \qquad \text{(for details see [TM], Chap. 10.2)} \end{aligned}$$

3.2.2 Kinetic energy and inertia tensor

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \mathbf{v}_i^2\Big|_{S_o} = \frac{1}{2}\sum_i \left(\dot{\mathbf{R}} + (\boldsymbol{\omega}\times\mathbf{r}_i)\Big|_{S_{f,b}}\right)^2 \\ &= \frac{1}{2}\sum_i m_i\Big\{\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\cdot(\boldsymbol{\omega}\times\mathbf{r}_i)\Big|_{S_{f,b}} + (\boldsymbol{\omega}\times\mathbf{r}_i)^2\Big|_{S_{f,b}}\Big\} \\ &= \underbrace{\frac{1}{2}M\dot{\mathbf{R}}^2}_{\| \atop T_{trans}} + \underbrace{\dot{\mathbf{R}}\cdot\left(\boldsymbol{\omega}\times\sum_i m_i\mathbf{r}_i\Big|_{S_{f,b}}\right)}_{=0 \text{ in} \atop \text{CM system}} + \underbrace{\frac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times\mathbf{r}_i)^2\Big|_{S_{f,b}}}_{\| \atop T_{rot}} \\ &= T_{trans} + T_{rot} \end{aligned}$$

$$
\begin{aligned}
T_{rot} &= \frac{1}{2}\sum_{i=1}^{N} m_i \Big(\boldsymbol{\omega}^2 \mathbf{r}_i^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_i)^2\Big)_{S_b} \quad \text{write it out in body system:} \\
&= \frac{1}{2}\sum_{i=1}^{N} m_i \Big\{ \sum_{j=1}^{3} \omega_j^2 \mathbf{r}_i^2 - \Big(\sum_{j=1}^{3}\omega_j x_i^{(j)}\Big)\Big(\sum_{k=1}^{3}\omega_k x_i^{(k)}\Big)\Big\} \\
&= \frac{1}{2}\sum_{i=1}^{N} m_i \sum_{j,k=1}^{3} \Big\{\mathbf{r}_i^2 \delta_{jk} - x_i^{(j)} x_i^{(k)}\Big\}\omega_j\omega_k \\
&= \frac{1}{2}\sum_{j,k=1}^{3} I_{jk}\omega_j\omega_k
\end{aligned}
$$

with $$I_{jk} = \sum_{i=1}^{N} m_i \Big\{\delta_{jk}\mathbf{r}_i^2 - x_i^{(j)} x_i^{(k)}\Big\} \quad \text{inertia tensor (inertia matrix)}$$

summary: $$\boxed{T_{rot} = \frac{1}{2}\,\boldsymbol{\omega}^T \underline{\underline{I}}\;\boldsymbol{\omega}}$$

3.2.3 Structure and properties of the inertia tensor

a) Continuous mass distributions

$$m_i = \Delta m_i = \rho(\mathbf{r}_i)\Delta V_i \quad \longrightarrow \quad \rho(\mathbf{r})\; d^3r = dm$$

$$M = \sum_{i=1}^{N} m_i \quad \longrightarrow \quad \int_V dm = \int_V \rho(\mathbf{r})\; d^3r$$

$$\boxed{I_{jk} = \int_V \rho(\mathbf{r})\Big\{\delta_{jk}\mathbf{r}^2 - x_j x_k\Big\}\; d^3r}$$

explicitly :

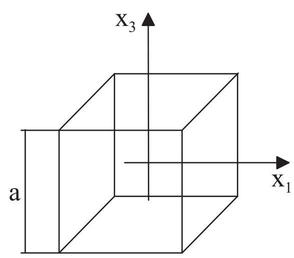
$$\left.\begin{aligned}
I_{11} &= \textstyle\int_V (x_2^2 + x_3^2)\rho(\mathbf{r})\; d^3r \\
I_{22} &= \textstyle\int_V (x_1^2 + x_3^2)\rho(\mathbf{r})\; d^3r \\
I_{33} &= \textstyle\int_V (x_1^2 + x_2^2)\rho(\mathbf{r})\; d^3r
\end{aligned}\right\} \text{“moments of inertia''}$$

$$\left.\begin{aligned} I_{12} = I_{21} &= -\int_V x_1 x_2 \rho(\mathbf{r})\ d^3r \\ I_{13} = I_{31} &= -\int_V x_1 x_3 \rho(\mathbf{r})\ d^3r \\ I_{23} = I_{32} &= -\int_V x_2 x_3 \rho(\mathbf{r})\ d^3r \end{aligned}\right\} \text{"products of inertia"}$$

note that $\underline{\underline{I}}$ is symmetric $(I_{jk} = I_{kj})$

b) Examples

(i) Homogeneous cube (I)

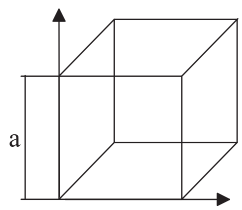


$$\rho(\mathbf{r}) = \begin{cases} \rho_0 = \frac{M}{a^3} & \mathbf{r} \in V \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} I_{11} = \frac{M}{a^3}\int_V (x_2^2 + x_3^2)\ d^3r &= \frac{M}{a^3}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} (x_2^2 + x_3^2)\ dx_1 dx_2 dx_3 \\ &= \frac{M}{a^2}\int_{-\frac{a}{2}}^{\frac{a}{2}} dx_2 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_3\ (x_2^2 + x_3^2) \\ &= \frac{M}{a^2}\Big(a\int_{-\frac{a}{2}}^{\frac{a}{2}} x_2^2\ dx_2 + a\int_{-\frac{a}{2}}^{\frac{a}{2}} x_3^2\ dx_3\Big) \\ &= \frac{M}{a}\Big(\frac{x_2^3}{3}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} + \frac{x_3^3}{2}\Big|_{-\frac{a}{2}}^{\frac{a}{2}}\Big) \\ &= \frac{1}{6}\ Ma^2\ =\ I_{22}\ =\ I_{33} \end{aligned}$$

$$\begin{aligned} I_{12} &= -\frac{M}{a^3}\int_V x_1 x_2\ dx_1 dx_2 dx_3 \\ &= -\frac{M}{a^3}\int_{-\frac{a}{2}}^{\frac{a}{2}} x_1\ dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} x_2\ dx_2 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_3 \\ &= -\frac{M}{4a^2}\ x_1^2\Big|_{-\frac{a}{2}}^{\frac{a}{2}} x_2^2\Big|_{-\frac{a}{2}}^{\frac{a}{2}} \\ &= 0\ =\ I_{13}\ =\ I_{23} \end{aligned}$$

(ii) Homogeneous cube (II)



$$
\begin{aligned}
I_{11} &= \frac{M}{a^3} \int_0^a \int_0^a \int_0^a \left(x_2^2 + x_3^2\right)\, dx_1 dx_2 dx_2 \\
&= \frac{M}{a} \left(\frac{x_2^3}{3}\bigg|_0^a + \frac{x_3^3}{2}\bigg|_0^a \right) \\
&= \frac{2}{3}\, Ma^2 \;=\; I_{22} \;=\; I_{33}
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= -\frac{M}{a^3} \int_0^a x_1 dx_1 \int_0^a x_2 dx_2 \int_0^a dx_3 \\
&= -\frac{M}{a^2} \left(\frac{1}{2}\, x_1^2\bigg|_0^a \right) \left(\frac{1}{2}\, x_2^2\bigg|_0^a \right) \\
&= -\frac{Ma^2}{4} \;=\; I_{13} \;=\; I_{23}
\end{aligned}
$$

c) Principal axes of inertia

Theorem: For any rigid body and any choice for the origin of the body system there exists a set of perpendicular ("principal") axes such that $I_{jk} = I_k \delta_{jk}$

'proof': $\underline{\underline{I}}$ is a (real) symmetric matrix $\Rightarrow$ can be diagonalized

diagonalization of $\underline{\underline{I}} \Longleftrightarrow \exists$ orthogonal matrix

$$
\underline{\underline{D}}^{-1} = \underline{\underline{D}}^{T} \quad \Longleftrightarrow \quad (\underline{\underline{D}}^{-1})_{kj} = D^{-1}_{kj} = D_{jk} \ ,
$$

such that $\underline{\underline{D}}^T \underline{\underline{I}}\, \underline{\underline{D}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad \Longleftrightarrow \quad \sum_{jk} D^T_{nj} I_{jk} D_{km} = \delta_{mn} I_n$

$$\Longleftrightarrow \quad \underline{\underline{I}}\,\underline{\underline{D}} = \underline{\underline{D}} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \Longleftrightarrow \sum_k I_{jk} D_{km} \begin{array}{l} = \sum_k D_{jk} I_k \delta_{km} \\ = D_{jm} I_m \\ = \sum_k \delta_{jk} D_{km} I_m \end{array}$$

$$\Longleftrightarrow \quad \boxed{\sum_k \Big(I_{jk} - I_m \delta_{jk} \Big) D_{km} = 0}$$

$$\Longleftrightarrow \quad \begin{pmatrix} I_{11} - I_m & I_{12} & I_{13} \\ I_{21} & I_{22} - I_m & I_{22} \\ I_{31} & I_{32} & I_{33} - I_m \end{pmatrix} \begin{pmatrix} D_{1m} \\ D_{2m} \\ D_{3m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad m = 1, 2, 3$$

condition for a nontrivial solution:

$$\det(I_{jk} - I_m \delta_{jk}) = 0$$

this (cubic) equation is called secular or characteristic equation.

<u>Diagonalization procedure:</u>

(i) Solve secular equation → obtain "eigenvalues" I_1, I_2, I_3

(ii) Find $\underline{\underline{D}}$ by inserting eigenvalues into the system of homogeneous equations above.

<u>Remarks:</u>

(i) I_k :"principal moments of inertia"; the axes of the corresponding body system are the principal axes

(ii) Transformation matrix $\underline{\underline{D}}$ characterizes a rotation in $\mathbb{R}^3$

(iii) The principal moments of inertia are consistent with the equation $I_k = \int_V \rho(\mathbf{r})(\mathbf{r}^2 - x_k^2)\ d^3r$

(iv) Change of $\underline{\underline{I}}$ when moving body system from CM to other origin: Steiner's parallel-axis theorem ([TM], Chap. 11.6)

(v) Kinetic energy

$$\text{if} \quad I_{jk} = I_k \delta_{jk} \quad \longrightarrow \quad T_{rot} = \frac{1}{2} \sum_k I_k \omega_k^2$$

3.2.4 Generalized coordinates and the Lagrangian

Lagrangian:

$$
\begin{aligned}
L &= T - U = T_{trans} + T_{rot} - U \\
&= \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}\sum_k I_k \omega_k^2 - U
\end{aligned}
$$

This form of the Lagrangian reinforces what is stated by Chasles' theorem: the translational motion of the rigid body can be described by the three CM coordinates, while we need three additional coordinates to characterize the rotational motion. Recall that we introduced a fixed inertial reference frame S_o, a fixed (i.e. non-rotating) reference system S_f with the CM as origin and a rotating body system S_b, the origin of which is also the CM. The translational motion of the body (the CM) is described by the transformation from S_o to S_f, while rotations correspond to the transformation from S_f to S_b:

$$S_o \overset{\text{translation}}{\longrightarrow} S_f \overset{\text{rotation}}{\longrightarrow} S_b$$

<u>Theorem:</u> any rotation can be described by a series of three rotations through the *Eulerian angles* (α, β, γ) about designated coordinate axes.

<u>Definition of the Euler angles:</u> (note that other conventions are also in use)

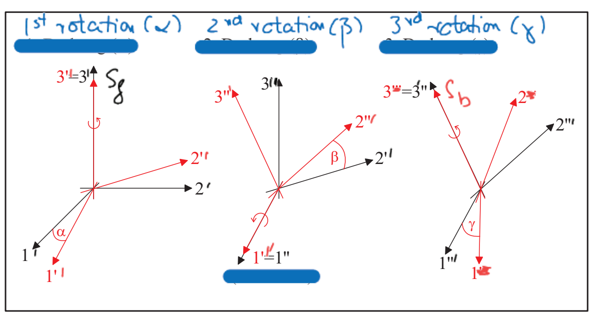


Figure 3.2: Definition of the Eulerian angles.

<u>Remarks</u>:

- rotations are characterized by orthogonal 3×3 matrices (det $\underline{\underline{D}} = 1$)

- rotation matrices form a nonabelian group, i.e., $\underline{\underline{D}}_1\underline{\underline{D}}_2 \neq \underline{\underline{D}}_2\underline{\underline{D}}_1$
- If we consider a vector $\mathbf{r}$ in 3D space, its components in the systems S_f and S_b are related by

$$\mathbf{r}_{S_b} \;=\; \underline{\underline{D}}_\gamma\, \underline{\underline{D}}_\beta\, \underline{\underline{D}}_\alpha\, \mathbf{r}_{S_f} \;\equiv\; \underline{\underline{D}}\, \mathbf{r}_{S_f},$$

  where $\underline{\underline{D}}_\alpha, \underline{\underline{D}}_\beta, \underline{\underline{D}}_\gamma$ are the rotation matrices that correspond to the rotations through the Euler angles. They are spelled out in Appendix A.5.

Using the rotation matrices $\underline{\underline{D}}_\alpha, \underline{\underline{D}}_\beta, \underline{\underline{D}}_\gamma$ one can determine the components of $\boldsymbol{\omega}$ in the body system (see Appendix A.5):

$$\begin{aligned}
\omega_1 &= \dot\alpha \sin\beta \sin\gamma + \dot\beta \cos\gamma \\
\omega_2 &= \dot\alpha \sin\beta \cos\gamma - \dot\beta \sin\gamma \\
\omega_3 &= \dot\alpha \cos\beta + \dot\gamma
\end{aligned}$$

This has to be inserted in the rotational kinetic energy expression in the Lagrangian. One obtains:

$$\begin{aligned}
L &= \frac{M}{2}\big(\dot X^2 + \dot Y^2 + \dot Z^2\big) \\
&\quad + \frac{1}{2}\Big\{ I_1\big(\dot\alpha \sin\beta \sin\gamma + \beta\cos\gamma\big)^2 + I_2\big(\dot\alpha \sin\beta\cos\gamma - \dot\beta\sin\gamma\big)^2 \\
&\quad + I_3\big(\dot\alpha\cos\beta + \dot\gamma\big)^2\Big\} - U\big(X, Y, Z, \alpha\beta\gamma\big) \\
&= L\big(XYZ, \alpha\beta\gamma; \dot X\dot Y\dot Z, \dot\alpha\dot\beta\dot\gamma\big)
\end{aligned}$$

3.2.5 Equations of motion

$$\text{translational motion}: \qquad \begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot X} = \frac{\partial L}{\partial X} &\iff M\ddot X = -\frac{\partial U}{\partial X} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot Y} = \frac{\partial L}{\partial Y} &\iff M\ddot Y = -\frac{\partial U}{\partial Y} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot Z} = \frac{\partial L}{\partial Z} &\iff M\ddot Z = -\frac{\partial U}{\partial Z}
\end{aligned}$$

example: rigid body in uniform gravitational field

$$U = \sum_i m_i g z_i \Big|_{S_o} = g \sum_i m_i z_i \Big|_{S_o} = MgZ = U(Z)$$

The EoMs for the translational motion are simple:

$$\ddot{X} = \ddot{Y} = 0, \quad \ddot{Z} = -g$$

If the body rotates (freely) about the CM (see Sec. 3.2.7) one has

$$\frac{\partial U}{\partial \alpha} = \frac{\partial U}{\partial \beta} = \frac{\partial U}{\partial \gamma} = 0$$

$\hookrightarrow$ rotational motion is not influenced by U, but is far from being trivial (this situation is called motion of a 'free top').

In the general case, one obtains a set of second-order nonlinear, coupled differential equations for the Euler angles α, β, γ for the description of the rotational motion (see Appendix A.5). These EoMs can be summarized as:

$$\boxed{\begin{array}{rcl} I_1\dot{\omega}_1 - \big(I_2 - I_3\big)\omega_2\omega_3 & = & N_1 \\ I_2\dot{\omega}_2 - \big(I_3 - I_1\big)\omega_1\omega_3 & = & N_2 \\ I_3\dot{\omega}_3 - \big(I_1 - I_2\big)\omega_1\omega_2 & = & N_3 \end{array}}$$

Euler's equations

N_i: components of torque in the body system. For the example with $U = MgZ$ all torque components N_i vanish.

3.2.6 Angular momentum

For details see [TM], Chap. 11.4 and 11.9
recap: angular momentum of a system of particles (cf. Sec. 1.2.2)

$$\begin{aligned} \mathbf{L} &= \sum_i (\mathbf{r}_i \times \mathbf{p}_i) \\ \dot{\mathbf{L}} &= \sum_i (\mathbf{r}_i \times \mathbf{F}_i) = \mathbf{N} \end{aligned}$$

One can show:

$$\begin{aligned}
\mathbf{L}\Big|_{S_o} &= \mathbf{L}_{CM} \quad + \mathbf{L}\Big|_{S_f} \\
&= M\big(\mathbf{R}\times\mathbf{V}\big) + \sum_i m_i\big(\mathbf{r}_i\times\mathbf{v}_i\big)\Big|_{S_f} \\
&= M\big(\mathbf{R}\times\mathbf{V}\big) + \sum_i m_i\big(\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i)\big)\Big|_{S_f}
\end{aligned}$$

The second term can be rewritten by decomposing the vectors in the body system and using the definition of the inertia tensor. One obtains

$$\mathbf{L}\Big|_{S_f} = \underline{\underline{I}}\,\boldsymbol{\omega}$$

Note that ths is the angular momentum measured by an observer in S_f who does **not** rotate with the rigid body (otherwise he would find $\mathbf{L}=0$). But the components of this representation of $\mathbf{L}$ refer to the body system S_b, i.e.

$$L_j\Big|_{S_b} = \sum_k I_{jk}\omega_k\Big|_{S_b}$$

<u>Remarks:</u>

(i) $L_j = I_j\ \omega_j$ in principal axes body system

(ii) in general $\mathbf{L} \not\parallel \boldsymbol{\omega} \quad \to$ motion of rigid bodies is complicated!

(iii) $T_{rot} = \frac{1}{2}\sum_{jk} I_{jk}\ \omega_k\ \omega_j = \frac{1}{2}\sum_j L_j\ \omega_j = \frac{1}{2}\ \mathbf{L}\cdot\boldsymbol{\omega}$

in principal axes system : $\quad T_{rot} = \dfrac{1}{2}\displaystyle\sum_j L_j\ \omega_j = \dfrac{1}{2}\sum_j \dfrac{L_j^2}{I_j}$

Note that this is the rotational kinetic energy measured by a non-rotating observer (otherwise she would measure $T_{rot}=0$).

If one rewrites $\dot{\mathbf{L}} = \mathbf{N}$ in body coordinates one reobtains Euler's equations

$$\begin{aligned}
\dot{L}_1\Big|_{S_b} &= I_1\dot{\omega}_1 - \big(I_2 - I_3\big)\omega_2\omega_3 = N_1 \\
\dot{L}_2\Big|_{S_b} &= I_2\dot{\omega}_2 - \big(I_3 - I_1\big)\omega_3\omega_1 = N_2 \\
\dot{L}_3\Big|_{S_b} &= I_3\dot{\omega}_3 - \big(I_1 - I_2\big)\omega_1\omega_2 = N_3
\end{aligned}$$

3.2.7 Applications: symmetric tops

top: a rigid body which is free to turn about a fixed point. The top is torque-free if the fixed point is the CM and $U = MgZ$ (see above).

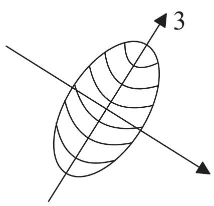
a) Free spherical top: $I_1 = I_2 = I_3 \equiv I$

$$\overset{\text{Euler:}}{\hookrightarrow} \quad \dot{\omega}_k = 0 \quad \longrightarrow \quad \omega_k(t) = \omega_k(0) = \text{const}$$

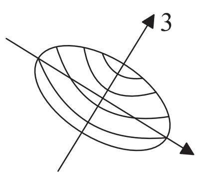
$\longrightarrow$ body rotates uniformly about fixed axis

b) Free symmetric top: $I_1 = I_2 = I,\ I_3 \neq I$

Two cases can be distinguished:



'prolate' top: $I > I_3$



'oblate' top: $I < I_3$

$$\text{Euler :} \quad \begin{aligned} I\dot{\omega}_1 + \big(I - I_3\big)\omega_2\omega_3 &= 0 \\ I\dot{\omega}_2 - \big(I_3 - I\big)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 \qquad\qquad &= 0 \quad \Longrightarrow \ \omega_3(t) = \omega_3 = \text{const} \end{aligned}$$

define: $\quad \Omega := \frac{I_3 - I}{I}\,\omega_3 \quad (= \text{const})$

$$\hookrightarrow \quad \left| \begin{aligned} \dot{\omega}_1 + \Omega\omega_2 = 0 \\ \\ \dot{\omega}_2 - \Omega\omega_1 = 0 \end{aligned} \right| \qquad \begin{aligned} (1) \\ \\ (2) \end{aligned}$$

decouple:

$$(1'): \qquad \ddot{\omega}_1 + \Omega\dot{\omega}_2 = 0 \quad \Longleftrightarrow \quad \dot{\omega}_2 = -\frac{1}{\Omega}\,\ddot{\omega}_1$$

$$(2'): \qquad -\frac{1}{\Omega}\,\ddot{\omega}_1 - \Omega\dot{\omega}_1 = 0 \quad \Longleftrightarrow \quad \ddot{\omega}_1 + \Omega^2\omega_1 = 0$$

$\hookrightarrow$ general solution : $\omega_1(t) = C_1 \cos\Omega t + C_2 \sin\Omega t$

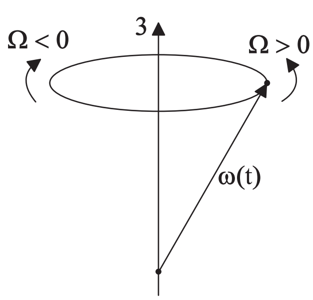
$$(1''): \qquad \frac{d}{dt}\Big(C_1\cos\Omega t + C_2 \sin\Omega t\Big) = -\Omega\omega_2$$
$$\Longleftrightarrow \quad \omega_2(t) = C_1 \sin\Omega t - C_2\cos\Omega t$$

$\lhd$ initial conditions: $\omega_1(0) = A\ , \quad \omega_2(0) = 0$

$$\boxed{\begin{array}{rcl} \omega_1(t) & = & A\cos\Omega t \\ \omega_2(t) & = & A\sin\Omega t \\ \omega_3(t) & = & \omega_3 \ = \ \text{const} \end{array}} \qquad \text{(components of } \boldsymbol{\omega} \text{ wrt } S_b)$$

$$|\boldsymbol{\omega}| = \sqrt{A^2 + \omega_3^2} = \text{const}$$

Visualization:

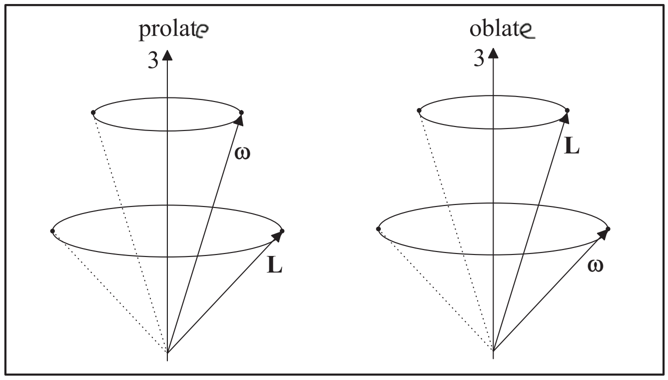


"free precession"

$$\mathbf{L} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} IA\cos\Omega t \\ IA\sin\Omega t \\ I_3\omega_3 \end{pmatrix}$$

$$\text{prolate}: \qquad \frac{L_3}{\sqrt{L_1^2 + L_2^2}} = \frac{I_3\omega_3}{IA} \ < \ \frac{\omega_3}{A} = \frac{\omega_3}{\sqrt{\omega_1^2+\omega_2^2}}$$

$$\text{oblate}: \qquad \frac{I_3\omega_3}{IA} \ > \ \frac{\omega_3}{A}$$



$$\boldsymbol{\omega}, \mathbf{L}, \mathbf{e}_3 \quad \text{are in a plane at all times} \quad (\text{show } \mathbf{e}_3 \cdot (\boldsymbol{\omega} \times \mathbf{L}) = 0)$$

$$T_{rot} = \frac{1}{2}\,\mathbf{L} \cdot \boldsymbol{\omega} = \frac{1}{2}\big(IA^2 + I_3\omega_3^2\big) = \text{const}$$

To better understand the motion of the torque-free top one has to look at it from the viewpoint of the inertial reference frame S_f. This can be done in two ways:

(i) A qualitative analysis starts from noting that the angular momentum in S_f is constant (since $\mathbf{N} = 0$). Our solution seems to contradict this, but the time dependence of the components of $\mathbf{L}$ is a consequence of decomposing it in S_b. When viewed from S_f, $\mathbf{L}$ is fixed, while $\boldsymbol{\omega}$ and the 3-axis of the rigid body are precessing. The motion of the rigid body can be visualized by considering two cones, the so-called body and space-fixed cones, see, e.g. [TM], Chap. 11.10 for details and illustrations.

(ii) The quantitative analysis involves the determination of the Eulerian angles. This can be done by comparing the explicit solution obtained for $\boldsymbol{\omega}$ in S_b with the general expression derived in Appendix A.5:

$$\boldsymbol{\omega} = \begin{pmatrix} A\cos\Omega t \\ A\sin\Omega t \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma \\ \dot{\alpha}\sin\beta\cos\gamma - \dot{\beta}\sin\gamma \\ \dot{\alpha}\cos\beta + \dot{\gamma} \end{pmatrix}$$

We know from the previous analysis that the angle between $\mathbf{L}$ and the 3-axis of S_b is constant (steady precession). If we choose the 3-axis of S_f to point along $\mathbf{L}$ this angle can be identified with the Euler angle $\beta = \beta_0 =$const.

$$\hookrightarrow \begin{pmatrix} A\cos\Omega t = \dot\alpha \sin\beta_0 \sin\gamma \\ A\sin\Omega t = \dot\alpha \sin\beta_0 \cos\gamma \\ \omega_3 = \dot\alpha\cos\beta_0 + \dot\gamma \end{pmatrix}$$

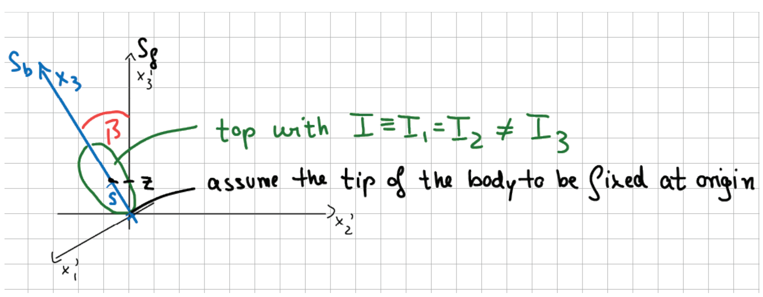
$$\hookrightarrow A^2 = \dot\alpha^2 \sin^2\beta_0 \quad\Rightarrow \dot\alpha = \pm\frac{A}{\sin\beta_0} = \text{const}$$

We obtain:

$$\begin{pmatrix} \alpha(t) = \pm\frac{A}{\sin\beta_0}t + \alpha_0 \\ \beta_0 = \tan^{-1}\left(\frac{IA}{I_3\omega_3}\right) \\ \gamma(t) = -\Omega t \pm \frac{\pi}{2} \end{pmatrix}$$

Interpretation: $\alpha(t)$ characterizes the steady precession of the symmetry axis of the rigid body about the 3-axis of S_f, β_0 the constant inclination of the symmetry axis, and $\gamma(t)$ the rotation of the rigid body about its symmetry axis.

c) The heavy symmetric top: $\mathbf{N} \neq 0$



(i) Potential energy

$$U = MgZ\Big|_{S_f} = MgS\cos\beta = U(\beta)$$

(ii) Kinetic energy

$$T_{\text{rot}} = \frac{1}{2}[I(\omega_1^2+\omega_2^2)+I_3\omega_3^2] = \ldots = \frac{1}{2}[I(\dot\alpha^2\sin^2\beta+\dot\beta^2)+I_3(\dot\alpha\cos\beta+\dot\gamma)^2]$$

(iii) Lagrangian

$$L = T_{\rm rot} - U = L(\beta, \dot\alpha, \dot\beta, \dot\gamma)$$

$\hookrightarrow \alpha, \gamma$ cyclic and

$$\begin{aligned} p_\alpha &= \frac{\partial L}{\partial \dot\alpha} = I\dot\alpha \sin^2\beta + I_3(\dot\alpha\cos^2\beta + \dot\gamma\cos\beta) = \text{const} \\ p_\gamma &= \frac{\partial L}{\partial \dot\gamma} = I_3(\dot\alpha\cos\beta + \dot\gamma) = I_3\omega_3 = \text{const} \\ \hookrightarrow p_\alpha &= I\dot\alpha\sin^2\beta + p_\gamma\cos\beta \end{aligned}$$

(iv) Analysis

Instead of working out the EoMs let's look at the (conserved) energy of the top and do something similar to what we have done for the central-force problem in Sec. 3.1:

$$E = H = T_{\rm rot} + U = \frac{1}{2}[I(\dot\alpha^2\sin^2\beta + \dot\beta^2) + I_3(\dot\alpha\cos\beta + \dot\gamma)^2] + MgS\cos\beta$$

We can eliminate $\dot\alpha$ and $\dot\gamma$ by using the constant momentum components and obtain

$$\begin{aligned} E &= \frac{1}{2}I\dot\beta^2 + \frac{1}{2}\frac{(p_\alpha - p_\gamma\cos\beta)^2}{I\sin^2\beta} + \frac{1}{2}\frac{p_\gamma^2}{I_3} + MgS\cos\beta \\ \Longleftrightarrow E - \frac{1}{2}\frac{p_\gamma^2}{I_3} &\equiv E' = \frac{1}{2}I\dot\beta^2 + U_{\rm eff}(\beta) \\ \text{with} \quad U_{\rm eff}(\beta) &= \frac{1}{2}\frac{(p_\alpha - p_\gamma\cos\beta)^2}{I\sin^2\beta} + MgS\cos\beta \end{aligned}$$

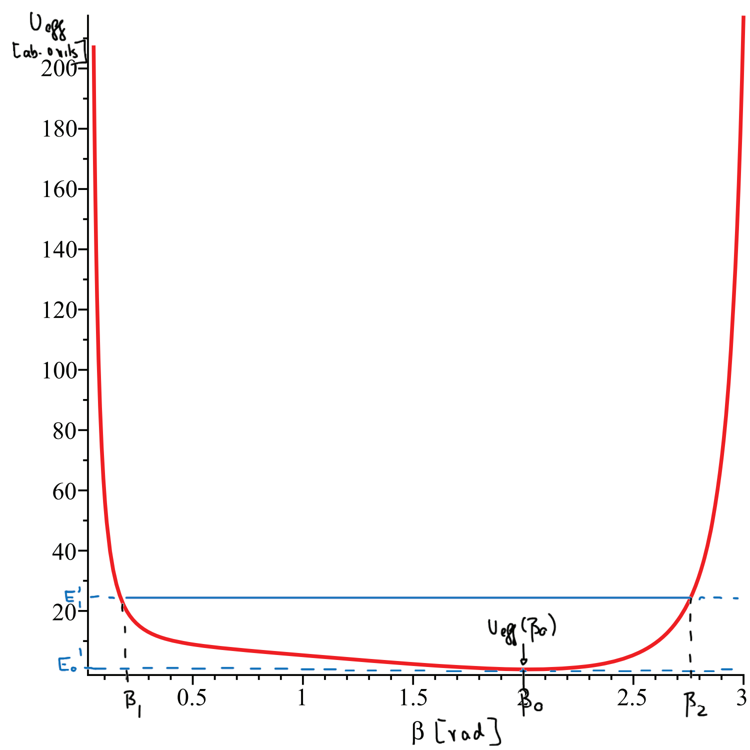
A formal solution is obtained from rewriting the equation for E' as a differential equation for β:

$$\dot\beta = \pm\sqrt{\frac{2}{I}(E' - U_{\rm eff}(\beta))} \quad \Rightarrow \quad \int dt = \pm \int \frac{d\beta}{\sqrt{\frac{2}{I}(E' - U_{\rm eff}(\beta))}}$$

Inversion yields $\beta(t)$, which in turn can be used to obtain $\alpha(t)$ and $\gamma(t)$ from integrating

$$\begin{aligned} \dot\alpha &= \frac{p_\alpha - p_\gamma\cos\beta(t)}{I\sin^2\beta(t)} \\ \dot\gamma &= \frac{p_\gamma}{I_3} - \dot\alpha\cos\beta(t) \end{aligned}$$

Unfortunately, these steps cannot be carried out analytically. A more qualitative discussion can be based on inspecting $U_{\rm eff}(\beta)$, which is a u-shaped function:



Observations

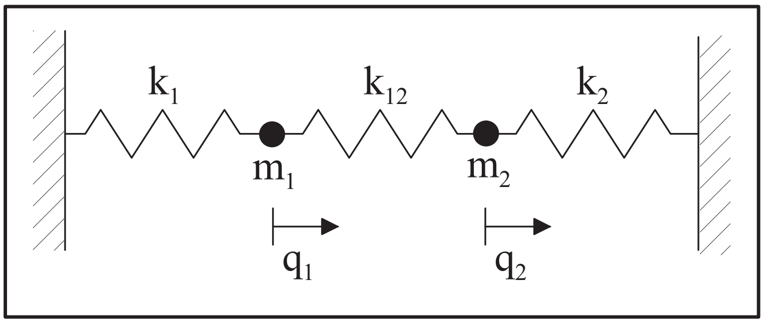
(i) Shallow minimum at β_0: If $E' = E'_0 = U_{\rm eff}(\beta_0)$ we have steady precession at a fixed inclination angle. This can be analyzed in more detail by inspecting $\left.\frac{dU_{\rm eff}}{d\beta}\right|_{\beta_0} = 0$.

(ii) For $E' = E'_1 > U_{\rm eff}(\beta_0)$ the inclination angle varies between β_1, β_2 with $0 < \beta_1 \leq \beta \leq \beta_2 < \pi$. This is called nutation. Depending on whether or not $\dot{\alpha} = \frac{p_\alpha - p_\gamma \cos\beta}{I \sin^2\beta}$ changes sign in the allowed β-interval three different types of motion can be distinguished (monotonic precession, looping motion, cusplike motion).

For details and illustrations see [TM], Chap. 11.11.

3.3 Coupled oscillations

3.3.1 An illustrative example: two coupled oscillators



- $T = \dfrac{m_1}{2}\,\dot{q}_1^2 + \dfrac{m_2}{2}\,\dot{q}_2^2$

- $U_{\text{ext}} = \dfrac{k_1}{2}\,q_1^2 + \dfrac{k_2}{2}\,q_2^2$

- $\bar{U} = \dfrac{k_{12}}{2}\big(q_1 - q_2\big)^2$

$$\hookrightarrow \quad \begin{aligned} L &= T - U_{\text{ext}} - \bar{U} \\ &= \frac{m_1}{2}\,\dot{q}_1^2 + \frac{m_2}{2}\,\dot{q}_2^2 - \frac{1}{2}\Big(k_1 q_1^2 + k_2 q_2^2 + k_{12}(q_1^2 - 2q_1 q_2 + q_2^2)\Big) \end{aligned}$$

Obtain equations of motion:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} &= m_1 \ddot{q}_1, \quad \frac{\partial L}{\partial q_1} = -k_1 q_1 - k_{12}(q_1 - q_2) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} &= m_2 \ddot{q}_2, \quad \frac{\partial L}{\partial q_2} = -k_2 q_2 + k_{12}(q_1 - q_2) \end{aligned}$$

$$\hookrightarrow \quad \begin{aligned} \ddot{q}_1 + \frac{k_1}{m_1} q_1 + \frac{k_{12}}{m_1}(q_1 - q_2) &= 0 \\ \ddot{q}_2 + \frac{k_2}{m_2} q_2 + \frac{k_{12}}{m_2}(q_2 - q_1) &= 0 \end{aligned}$$

- special case #1: $k_{12} = 0$ (no coupling)

$$q_\lambda(t) = A_\lambda \cos(\omega_\lambda t + \delta_\lambda),\ \lambda = 1, 2; \quad \omega_\lambda = \sqrt{\frac{k_\lambda}{m_\lambda}}$$

- special case #2: $m_1 = m_2 \equiv m, k_1 = k_2 \equiv k$

$$\begin{aligned} \hookrightarrow \quad \ddot{q}_1 + \frac{k + k_{12}}{m} q_1 - \frac{k_{12}}{m} q_2 &= 0 \\ \ddot{q}_2 + \frac{k + k_{12}}{m} q_2 - \frac{k_{12}}{m} q_1 &= 0 \end{aligned}$$

The form of the (internal) potential energy suggests the following transformation to another set of coordinates Q_λ:

$$\begin{aligned} Q_1 &= \frac{1}{2}(q_1 + q_2) \\ Q_2 &= \frac{1}{2}(q_1 - q_2) \end{aligned}$$

Inserting the inverse transformation into the equations of motion and adding and subtracting them yields the uncoupled equations and solutions

$$\begin{aligned} \ddot{Q}_1 + \frac{k}{m} Q_1 &= 0, \quad Q_1(t) = A_1 \cos(\Omega_1 t + \delta_1), \quad \Omega_1 = \sqrt{\frac{k}{m}} \\ \ddot{Q}_2 + \frac{k + 2k_{12}}{m} Q_2 &= 0, \quad Q_2(t) = A_2 \cos(\Omega_2 t + \delta_2), \quad \Omega_2 = \sqrt{\frac{k + 2k_{12}}{m}} \end{aligned}$$

so that we obtain

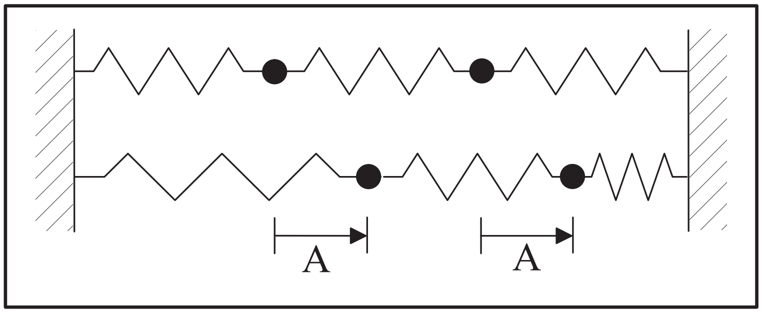
$$\begin{aligned} q_1(t) = A_1 \cos(\Omega_1 t + \delta_1) + A_2 \cos(\Omega_2 t + \delta_2) \\ q_2(t) = A_1 \cos(\Omega_1 t + \delta_1) - A_2 \cos(\Omega_2 t + \delta_2) \end{aligned}$$

as general solutions.

<u>Discussion</u>

(i) Q_1, Q_2: "normal coordinates", Ω_1, Ω_2: "normal frequencies"

(ii) pick initial conditions: $q_1(0) = q_2(0) = A, \dot{q}_1(0) = \dot{q}_2(0) = 0$

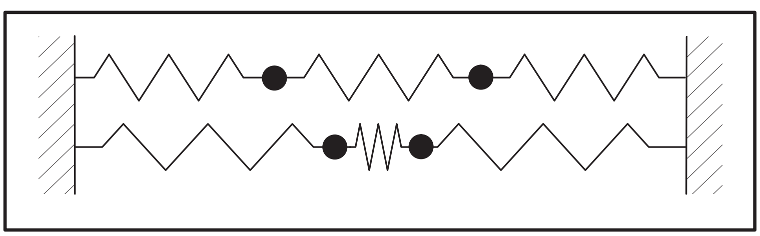


$$\begin{aligned}
q_1(0) &= A = A_1 \cos\delta_1 + A_2 \cos\delta_2 \\
q_2(0) &= A = A_1 \cos\delta_1 - A_2 \cos\delta_2 \\
\dot{q}_1(0) &= 0 = -A_1\Omega_1 \sin\delta_1 - A_2\Omega_2 \sin\delta_2 \\
\dot{q}_2(0) &= 0 = -A_1\Omega_1 \sin\delta_1 + A_2\Omega_2 \sin\delta_2
\end{aligned}$$

$$\hookrightarrow \quad A_1 = A \ , \quad A_2 = 0 \ , \quad \delta_1 = 0$$

$\hookrightarrow$ $\boxed{q_1(t) = A\cos\Omega_1 t = q_2(t)}$ $\rightarrow$ symmetrical normal mode

(iii) pick initial conditions: $q_1(0) = -q_2(0) = A, \dot{q}_1(0) = \dot{q}_2(0) = 0$



$\hookrightarrow$ $\boxed{q_1(t) = A\cos\Omega_2 t = -q_2(t)}$ $\rightarrow$ antisymmetrical normal mode

(iv) Note that $\Omega_2 > \Omega_1$ reflects the fact that more energy is stored in the antisymmetrical than in the symmetrical mode.

(v) Consider a variant: hold m_2 fixed, i.e., introduce the constraint $q_2 = 0$.

$$\overset{EoM}{\hookrightarrow} \quad \ddot{q}_1 + \frac{k + k_{12}}{m} q_1 = 0$$

general solution:

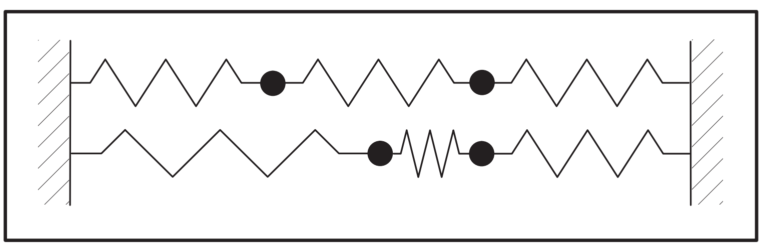
$$q_1(t) = A_1 \cos(\omega_0 t + \delta_1), \quad \omega_0 = \sqrt{\frac{k + k_{12}}{m}}$$

Similarly, one finds for $q_1 = 0$:

$$q_2(t) = A_2 \cos(\omega_0 t + \delta_2)$$

Observation: the 'common frequency' ω_0 splits into two frequencies according to $\Omega_1 < \omega_0 < \Omega_2$ if coupling is turned on.

(vi) pick initial conditions: $q_1(0) = A, q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$



solution:

$$\boxed{\begin{aligned} q_1(t) &= \frac{A}{2}\Big(\cos\Omega_1 t + \cos\Omega_2 t\Big) = A\cos\frac{(\Omega_1 + \Omega_2)t}{2}\cos\frac{(\Omega_1 - \Omega_2)t}{2} \\ q_2(t) &= \frac{A}{2}\Big(\cos\Omega_1 t - \cos\Omega_2 t\Big) = -A\sin\frac{(\Omega_1 + \Omega_2)t}{2}\sin\frac{(\Omega_1 - \Omega_2)t}{2} \end{aligned}}$$

$\rightarrow$ beats

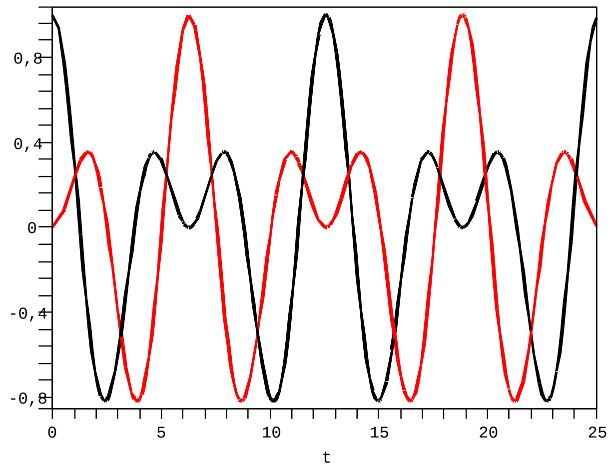


Figure 3.3: Beats for $\Omega_1 = 1$, $\Omega_2 = 1.5$; black curve: $q_1(t)$, red curve: $q_2(t)$

special case: $k_{12} \ll k$ (weak coupling)

$$\left.\begin{aligned} q_1(t) &\approx A\cos\Delta t\cos\omega_0 t \\ q_2(t) &\approx A\sin\Delta t\sin\omega_0 t \end{aligned}\right\} \quad \Delta = \frac{\Omega_1}{2}\frac{k_{12}}{k} \ll \Omega_1, \omega_0$$

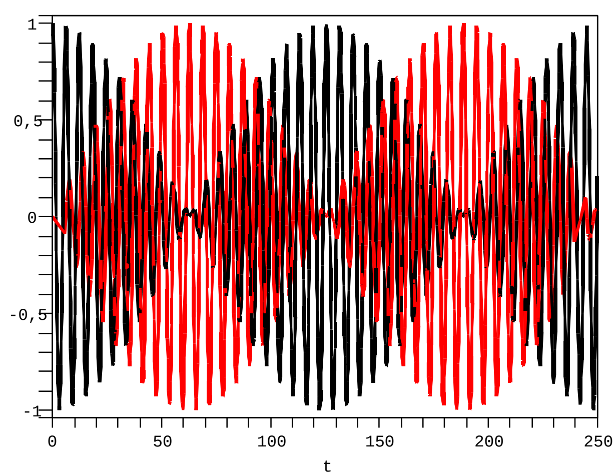


Figure 3.4: Beats for $\Omega_1 = 1$, $\Omega_2 = 1.05$; black curve: $q_1(t)$, red curce: $q_2(t)$

3.3.2 Lagrangian and equations of motion for coupled oscillations: general case

Let us characterize the systems of interest by the following properties:

(i) At most holonomic constraints

(ii) Time-independent transformations $x_i \longleftrightarrow q_\mu$

(iii) Conservative forces

(iv) System is in vicinity of stable equilibrium

$$\hookrightarrow \quad L = T - U_{\text{ext}} - \bar{U} \;\equiv\; T - U$$
$$H = T + U = E = \text{const}$$

- Taylor-expand $U(\mathbf{q})$ about equilibrium configuration $\mathbf{q}^0 = \mathbf{0}$

$$\begin{aligned} U(\mathbf{q}) &\approx \underbrace{U(0)}_{=0} + \sum_\mu \underbrace{\frac{\partial U}{\partial q_\mu}\Big|_0}_{=0} q_\mu + \frac{1}{2}\sum_{\mu\nu} \frac{\partial^2 U}{\partial q_\mu \partial q_\nu}\Big|_0 q_\mu q_\nu + ... \\ &\approx \frac{1}{2}\sum_{\mu\nu} U_{\mu\nu} q_\mu q_\nu \;, \qquad\qquad \left(U_{\mu\nu} := \frac{\partial^2 U}{\partial q_\mu \partial q_\nu}\Big|_0 = U_{\nu\mu}\right) \end{aligned}$$

- $T = \sum_i \frac{m_i}{2}\dot{x}_i^2 \quad \text{with} \qquad \dot{x}_i \;=\; \sum_\mu \frac{\partial x_i}{\partial q_\mu}\,\dot{q}_\mu$

$$\hookrightarrow \quad T \;=\; \frac{1}{2}\sum_{\mu\nu} T_{\mu\nu}(\mathbf{q})\dot{q}_\mu \dot{q}_\nu$$

$$\text{with} \quad T_{\mu\nu}(\mathbf{q}) \;=\; \sum_i m_i \frac{\partial x_i}{\partial q_\mu}\frac{\partial x_i}{\partial q_\nu} \;=\; T_{\nu\mu}(\mathbf{q})$$

Taylor-expand: $\qquad T_{\mu\nu}(\mathbf{q}) \;\approx\; T_{\mu\nu}(0) + \sum_\lambda \frac{\partial T_{\mu\sigma}}{\partial q_\lambda}\Big|_0 q_\lambda \;\approx\; T_{\mu\nu}(0) \;\equiv\; T_{\mu\nu} = T_{\nu\mu}$

(0^{th} order for $T_{\mu\nu}(\mathbf{q})$ is consistent with second order for U($\mathbf{q}$))

$$\hookrightarrow \quad L = \frac{1}{2}\sum_{\mu\nu}\Big(T_{\mu\nu}\dot{q}_\mu\dot{q}_\nu - U_{\mu\nu}q_\mu q_\nu\Big) = \frac{1}{2}\Big(\dot{\mathbf{q}}^T \underline{\underline{T}}\dot{\mathbf{q}} - \mathbf{q}^T\underline{\underline{U}}\mathbf{q}\Big)$$

note that the matrices $\underline{\underline{T}}$ and $\underline{\underline{U}}$ are symmetrical

Lagrangian EoMs:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}_\lambda} = \frac{\partial T}{\partial \dot{q}_\lambda} &= \frac{1}{2}\sum_{\mu\nu} T_{\mu\nu}\Big(\delta_{\mu\lambda}\dot{q}_\nu + \delta_{\nu\lambda}\dot{q}_\mu\Big) \\
&= \frac{1}{2}\sum_{\nu} T_{\lambda\nu}\dot{q}_\nu + \frac{1}{2}\sum_{\mu} T_{\mu\lambda}\dot{q}_\mu \\
&= \sum_{\mu} T_{\lambda\mu}\dot{q}_\mu \\
-\frac{\partial L}{\partial q_\lambda} = \frac{\partial U}{\partial q_\lambda} &= \ ... \ = \sum_{\mu} U_{\lambda\mu} q_\mu
\end{aligned}$$

$$\boxed{\begin{aligned}
\Longrightarrow \quad \sum_{\mu}\Big(T_{\lambda\mu}\ddot{q}_\mu + U_{\lambda\mu}q_\mu\Big) &= 0\ , \qquad \lambda = 1, ..., n \quad \text{for } n \text{ dofs} \\
\underline{\underline{T}}\ddot{\mathbf{q}} + \underline{\underline{U}}\mathbf{q} &= 0
\end{aligned}}$$

explicitly:

$$\begin{pmatrix} T_{11} & .... & T_{1n} \\ \vdots & & \\ T_{n1} & .... & T_{nn} \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{pmatrix} + \begin{pmatrix} U_{11} & .... & U_{1n} \\ \vdots & & \\ U_{n1} & .... & U_{nn} \end{pmatrix}\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

3.3.3 Solution of the EoMs

a) Simple situation: uncoupled oscillators

$$\text{characterized by} \quad T_{\lambda\mu} = \delta_{\lambda\mu}T_\lambda\ , \quad U_{\lambda\mu} = \delta_{\lambda\mu}U_\lambda$$

$$\overset{\text{EoMs}}{\hookrightarrow} \qquad \ddot{q}_\lambda + \frac{U_\lambda}{T_\lambda}\, q_\lambda = 0\ , \qquad \lambda = 1, ..., n$$

$$\text{solution :} \qquad q_\lambda(t) = A_\lambda \cos(\omega_\lambda t + \delta_\lambda)\ , \qquad \omega_\lambda = \sqrt{\frac{U_\lambda}{T_\lambda}}$$

b) Intermediate situation: diagonal $\underline{\underline{T}}$-matrix

characterized by $\quad T_{\lambda\mu} = T_\lambda \delta_{\lambda\mu}$

$$\overset{\text{EoMs}}{\hookrightarrow} \qquad T_\lambda \ddot{q}_\lambda + \sum_\mu U_{\lambda\mu} q_\mu = 0$$

transformation (i): $\quad \tilde{q}_\lambda = \sqrt{T_\lambda}\, q_\lambda$

$$\overset{\text{EoMs}}{\hookrightarrow} \qquad \ddot{\tilde{q}}_\lambda + \sum_\mu S_{\lambda\mu} \tilde{q}_\mu = 0$$

$$\text{with} \quad S_{\lambda\mu} = \frac{U_{\lambda\mu}}{\sqrt{T_\lambda T_\mu}} \quad \text{“dynamical matrix''}$$

$\underline{\underline{S}}$ is symmetric, positive semi-definite matrix, i.e.,

$$S_{\lambda\mu} = S_{\mu\lambda} \ , \quad \boldsymbol{\xi}^T \underline{\underline{S}}\, \boldsymbol{\xi} \geq 0 \quad \forall \ \ \boldsymbol{\xi} = \{\xi_1 ... \xi_n\} \neq \{0...0\}$$

$\Longrightarrow$ $\underline{\underline{S}}$ can be diagonalized; eigenvalues are real (since $\underline{\underline{S}}$ is symmetric) and non-negative (since $\underline{\underline{S}}$ is positive semi-definite)

That is, there is an orthogonal transformation matrix $\underline{\underline{V}}$ ($\underline{\underline{V}}^{-1} = \underline{\underline{V}}^T$) such that:

$$\underline{\underline{V}}^T \underline{\underline{S}}\, \underline{\underline{V}} = \begin{pmatrix} \Omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \Omega_n^2 \end{pmatrix} = \underline{\underline{\Omega}}^2 \ , \qquad (\Omega_i^2 \geq 0)$$

This motivates transformation (ii):

$$\begin{aligned} \mathbf{Q} = \underline{\underline{V}}^T\ \tilde{\mathbf{q}} \ &\Longleftrightarrow\ \tilde{\mathbf{q}} = \underline{\underline{V}}\ \mathbf{Q} \\ \overset{\text{EoMs}}{\hookrightarrow} \quad \underline{\underline{V}}\ \ddot{\mathbf{Q}} \quad &+ \quad \underline{\underline{S}}\ \underline{\underline{V}}\ \mathbf{Q} = 0 \\ \underline{\underline{V}}^T \underline{\underline{V}}\ \ddot{\mathbf{Q}} \quad &+ \quad \underline{\underline{V}}^T \underline{\underline{S}}\ \underline{\underline{V}}\ \mathbf{Q} = 0 \end{aligned}$$

$$\Longleftrightarrow \qquad \boxed{\quad \ddot{\mathbf{Q}} + \underline{\underline{\Omega}}^2\ \mathbf{Q} = 0 \quad}$$

$$\Longleftrightarrow \quad \boxed{\ddot{Q}_\lambda + \Omega_\lambda^2 Q_\lambda = 0} \quad \lambda = 1, ..., n$$

$$\text{solutions :} \quad \underset{\text{normal coordinates}}{\underset{\uparrow}{Q_\lambda(t)}} = \underset{\text{normal frequencies}}{\underset{\uparrow}{A_\lambda \cos(\Omega_\lambda t + \delta_\lambda)}}$$

Recipe:

(i) Set up and diagonalize $\underline{\underline{S}}$: solve secular equation

$$\det(S_{\mu\nu} - \Omega_\Lambda^2 \delta_{\mu\nu}) = 0$$

$\longrightarrow$ obtain

$$0 \leq \Omega_1^2 < \Omega_2^2 < ... < \Omega_n^2$$

(ii) Insert Ω_λ^2 into matrix equations to obtain $\underline{\underline{V}}$:

$$\sum_\nu (S_{\mu\nu} - \Omega_\lambda^2 \delta_{\mu\nu}) V_{\nu\lambda} = 0$$

(iii) Obtain

$$\begin{aligned} q_\mu(t) &= \frac{1}{\sqrt{T_\mu}} \tilde{q}_\mu(t) = \sum_\lambda \frac{V_{\mu\lambda}}{\sqrt{T_\mu}} Q_\lambda(t) \\ &= \sum_\lambda \frac{V_{\mu\lambda}}{\sqrt{T_\mu}} A_\lambda \cos(\Omega_\lambda t + \delta_\lambda) \ , \qquad \mu = 1, ..., n \end{aligned}$$

Applications

example (i): two coupled oscillators again (cf. Sec. 3.3.1)

The Lagrangian can be written as:

$$\hookrightarrow \quad L = \frac{1}{2} \left\{ (\dot{q}_1, \dot{q}_2) \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} - (q_1, q_2) \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\}$$

diagonalize $S_{\mu\nu} = \frac{U_{\mu\nu}}{\sqrt{T_\mu T_\nu}}$

$$\longrightarrow \quad \begin{vmatrix} \frac{k_1+k_{12}}{m_1} - \Omega^2 & -\frac{k_{12}}{\sqrt{m_1 m_2}} \\ -\frac{k_{12}}{\sqrt{m_1 m_2}} & \frac{k_2+k_{12}}{m_2} - \Omega^2 \end{vmatrix} = 0$$

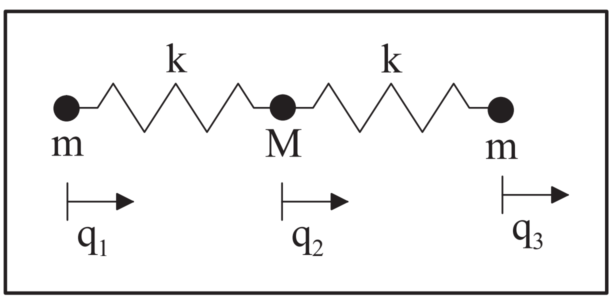
$$\Longleftrightarrow \quad \Big(\frac{k_1+k_{12}}{m_1} - \Omega^2\Big)\Big(\frac{k_2+k_{12}}{m_2} - \Omega^2\Big) - \frac{k_{12}^2}{m_1 m_2} = 0$$

$$\Longrightarrow \quad \Omega_{1,2}^2 = +\frac{1}{2}\Big(\frac{k_1+k_{12}}{m_1} + \frac{k_2+k_{12}}{m_2}\Big) \pm \frac{1}{2}\left[\Big(\frac{k_1+k_{12}}{m_1} - \frac{k_2+k_{12}}{m_2}\Big)^2 + \frac{4k_{12}^2}{m_1 m_2}\right]^{\frac{1}{2}}$$

for the special case $m_1 = m_2 \equiv m$, $k_1 = k_2 \equiv k$ one reobtains

$$\Omega_1 = \sqrt{\frac{k}{m}}\ , \qquad \Omega_2 = \sqrt{\frac{k+2k_{12}}{m}}$$

example (2): longitudinal vibrations of a linear triatomic molecule



- $T = \frac{m}{2}\Big(\dot{q}_1^2 + \dot{q}_3^2\Big) + \frac{M}{2}\,\dot{q}_2^2 = \frac{1}{2}\sum_{\mu\nu} T_{\mu\nu}\dot{q}_\mu \dot{q}_\nu$

$$\hookrightarrow \underline{\underline{T}} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

- $$\begin{aligned} U &= \frac{k}{2}(q_1-q_2)^2+\frac{k}{2}(q_3-q_2)^2 \\ &= \frac{k}{2}\Big(q_1^2+2q_2+q_3^2-2q_1q_2-2q_2q_3\Big) \\ &= \frac{1}{2}\sum_{\mu\nu} U_{\mu\nu}q_\mu q_\nu \end{aligned}$$

$$\hookrightarrow \underline{\underline{U}} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

- $$S_{\mu\nu} = \frac{U_{\mu\nu}}{\sqrt{T_\mu T_\nu}}$$

$$\hookrightarrow \underline{\underline{S}} = \begin{pmatrix} \frac{k}{m} & -\frac{k}{\sqrt{mM}} & 0 \\ -\frac{k}{\sqrt{mM}} & \frac{2k}{m} & -\frac{k}{\sqrt{mM}} \\ 0 & -\frac{k}{\sqrt{mM}} & \frac{k}{m} \end{pmatrix}$$

- solve characteristic equation

$$\begin{vmatrix} \frac{k}{m}-\Omega^2 & -\frac{k}{\sqrt{mM}} & 0 \\ -\frac{k}{\sqrt{mM}} & \frac{2k}{m}-\Omega^2 & -\frac{k}{\sqrt{mM}} \\ 0 & -\frac{k}{\sqrt{mM}} & \frac{k}{m}-\Omega^2 \end{vmatrix} = 0$$

$$\begin{aligned} \Longleftrightarrow \quad & \Big(\frac{k}{m}-\Omega^2\Big)\Big\{\Big(\frac{2k}{M}-\Omega^2\Big)\Big(\frac{k}{m}-\Omega^2\Big)-\frac{k^2}{mM}\Big\} \\ & \qquad -\frac{k}{\sqrt{mM}}\Big(\frac{k}{\sqrt{mM}}\Big(\frac{k}{m}-\Omega^2\Big)\Big) = 0 \\ \Longleftrightarrow \quad & \Big(\frac{k}{m}-\Omega^2\Big)\Big\{\Big(\frac{2k}{M}-\Omega^2\Big)\Big(\frac{k}{m}-\Omega^2\Big)-\frac{2k^2}{mM}\Big\} = 0 \end{aligned}$$

$$\Longleftrightarrow \quad \Omega_2^2 = \frac{k}{m}$$

$$\begin{aligned}
\frac{2k^2}{mM} - \Omega^2\Big(\frac{k}{m} + \frac{2k}{M}\Big) + \Omega^4 - \frac{2k^2}{mM} &= 0 \\
\iff \Omega^2\Big(\Omega^2 - \frac{k}{m}\Big(1 + \frac{2m}{M}\Big)\Big) &= 0
\end{aligned}$$

$$\Longrightarrow \quad \Omega_1^2 = 0$$

$$\Omega_3^2 = \frac{k}{m}\Big(1 + \frac{2m}{M}\Big)$$

$$\hookrightarrow \quad \boxed{\Omega_1 = 0\ , \quad \Omega_2 = \sqrt{\frac{k}{m}}\ , \quad \Omega_3 = \sqrt{\frac{k}{mM}\Big(2m + M\Big)}}$$

- transformation matrix (= eigenvectors)

$$\sphericalangle \qquad \begin{pmatrix} \frac{k}{m} - \Omega_\lambda^2 & -\frac{k}{\sqrt{mM}} & 0 \\ -\frac{k}{\sqrt{mM}} & \frac{2k}{M} - \Omega_\lambda^2 & -\frac{k}{\sqrt{mM}} \\ 0 & -\frac{k}{\sqrt{mM}} & \frac{k}{m} - \Omega_\lambda^2 \end{pmatrix} \begin{pmatrix} V_{1\lambda} \\ V_{2\lambda} \\ V_{3\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Omega_1 = 0$:

$$\left| \begin{array}{r} \frac{k}{m}V_{11} - \frac{k}{\sqrt{mM}}V_{21} = 0 \\ -\frac{k}{\sqrt{mM}}V_{11} + \frac{2k}{M}V_{21} - \frac{k}{\sqrt{mM}}V_{31} = 0 \\ -\frac{k}{\sqrt{mM}}V_{21} + \frac{k}{m}V_{31} = 0 \end{array} \right| \Longrightarrow \quad \underline{V}_{\,1} = \alpha \begin{pmatrix} \sqrt{\frac{m}{M}} \\ 1 \\ \sqrt{\frac{m}{M}} \end{pmatrix}$$

$\Omega_2^2 = \frac{k}{m}$:

$$\left| \begin{array}{r} -\frac{k}{\sqrt{mM}}V_{22} = 0 \\ -\frac{k}{\sqrt{mM}}V_{12} + k(\frac{2}{M} - \frac{1}{m})V_{22} - \frac{k}{\sqrt{mM}}V_{32} = 0 \\ -\frac{k}{\sqrt{mM}}V_{22} = 0 \end{array} \right| \Longrightarrow \quad \underline{V}_{\,2} = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\Omega_3^2 = \frac{k}{mM}(2m + M)$:

$$\left| \begin{array}{r} -\frac{2k}{M}V_{13} - \frac{k}{\sqrt{mM}}V_{23} = 0 \\ -\frac{k}{\sqrt{mM}}V_{13} - \frac{k}{m}V_{23} - \frac{k}{\sqrt{mM}}V_{33} = 0 \\ -\frac{k}{\sqrt{mM}}V_{23} - \frac{2k}{m}V_{33} = 0 \end{array} \right| \Longrightarrow \quad \underline{V}_{\,3} = \gamma \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{M}{m}} \\ 1 \\ -\frac{1}{2}\sqrt{\frac{M}{m}} \end{pmatrix}$$

normalization:

$$\Longrightarrow \quad \alpha^2(1+\frac{2m}{M}) = 1 \quad \Longleftrightarrow \quad \alpha = \sqrt{\frac{M}{M+2m}}$$

$$2\beta^2 = 1 \quad \Longleftrightarrow \quad \beta = \frac{1}{\sqrt{2}}$$

$$\gamma^2(1+\frac{1}{2}\frac{M}{m}) = 1 \quad \Longleftrightarrow \quad \gamma = \sqrt{\frac{2m}{M+2m}}$$

$$\Longrightarrow \quad \underline{\underline{V}} = \begin{pmatrix} \sqrt{\frac{m}{M+2m}} & \frac{1}{\sqrt{2}} & -\sqrt{\frac{M}{2(M+2m)}} \\ \sqrt{\frac{M}{M+2m}} & 0 & \sqrt{\frac{2m}{M+2m}} \\ \sqrt{\frac{m}{M+2m}} & -\frac{1}{\sqrt{2}} & -\sqrt{\frac{M}{2(M+2m)}} \end{pmatrix}$$

- general solution

$$\begin{aligned} q_1(t) &= \frac{1}{\sqrt{M+2m}}\, Q_1(t) + \frac{1}{\sqrt{2m}}\, Q_2(t) - \sqrt{\frac{M}{2m(M+2m)}}\, Q_3(t) \\ q_2(t) &= \frac{1}{\sqrt{M+2m}}\, Q_1(t) + \sqrt{\frac{2m}{M(M+2m)}}\, Q_3(t) \\ q_3(t) &= \frac{1}{\sqrt{M+2m}}\, Q_1(t) - \frac{1}{\sqrt{2m}}\, Q_2(t) - \sqrt{\frac{M}{2m(M+2m)}}\, Q_3(t) \end{aligned}$$

$$\text{with} \quad \begin{aligned} Q_1(t) &= A_1 t + B_1 \quad (\ddot{Q}_1 = 0) \\ Q_2(t) &= A_2 \cos(\Omega_2 t + \delta_2) \\ Q_3(t) &= A_3 \cos(\Omega_3 t + \delta_3) \end{aligned}$$

- initial conditions to bring out normal oscillations

case (i) :

$$\begin{aligned} q_1(0) &= q_2(0) = q_3(0) = 0 \\ \dot{q}_1(0) &= \dot{q}_2(0) = \dot{q}_3(0) = v_0 \end{aligned}$$

$$\Longrightarrow \quad q_1(t) = q_2(t) = q_3(t) = v_0 t$$

$\hookrightarrow$ translation of entire molecule

$$\text{case (ii)} : \qquad \begin{aligned} q_1(0) &= A = -q_3(0)\ , \quad q_2(0) = 0 \\ \dot q_1(0) &= \dot q_2(0) = \dot q_3(0) = 0 \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} q_1(t) &= A\cos\Omega_2 t = -q_3(t) \\ q_2(t) &= 0 \end{aligned}$$

$$\text{case (iii)} : \qquad \begin{aligned} q_1(0) &= q_3(0) = -A\ , \quad q_2(0) = \frac{2m}{M}A \\ \dot q_1(0) &= \dot q_2(0) = \dot q_3(0) = 0 \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} q_1(t) &= -A\cos\Omega_3 t = q_3(t) \\ q_2(t) &= \frac{2m}{M}A\cos\Omega_3 t \end{aligned}$$

- visualization of normal models

$$\begin{array}{llll} \Omega_1 : & \bullet\longrightarrow \quad \bullet\longrightarrow \quad \bullet\longrightarrow & & \text{translation} \\ \Omega_2 : & \bullet\longrightarrow \quad\ \ \bullet \quad\ \ \longleftarrow\bullet & & \\ \Omega_3 : & \longleftarrow\bullet \quad \bullet\longrightarrow \quad \longleftarrow\bullet & & \end{array}$$

c) General case (just briefly)

- EoMs : $\displaystyle \sum_\nu \Big(T_{\mu\nu}\ddot q_\nu + U_{\mu\nu}q_\nu\Big) = 0\,, \qquad \mu = 1, ..., n$

- ansatz (motivated by case b)):

$$\begin{aligned} q_\nu(t) &= \sum_\lambda V_{\nu\lambda}Q_\lambda(t) \qquad \text{with} \quad Q_\lambda(t) = A_\lambda\cos(\Omega_\lambda t + \delta_\lambda) \\ \hookrightarrow \qquad \ddot q_\nu &= \sum_\lambda V_{\nu\lambda}\ddot Q_\lambda \\ &= -\sum_\lambda V_{\nu\lambda}\Omega_\lambda^2 Q_\lambda \end{aligned}$$

$$\overset{\text{insert}}{\hookrightarrow} \qquad \sum_{\nu\lambda}\Big(U_{\mu\nu} - \Omega_\lambda^2 T_{\mu\nu}\Big)V_{\nu\lambda}Q_\lambda = 0$$

since Q_λ are linearly independent, one has:

$$\sum_\nu \Big(U_{\mu\nu} - \Omega_\lambda^2 T_{\mu\nu}\Big) V_{\nu\lambda} = 0 \qquad (*)$$

- This set of equations has a nontrivial solution if:

$$\det \Big(U_{\mu\nu} - \Omega_\lambda^2 T_{\mu\nu}\Big) = 0$$

$\Longrightarrow$ obtain $0 \leq \Omega_1^2 < ... < \Omega_n^2$

- Insert Ω_λ^2 into $(*)$ to obtain the transformation matrix $\underline{\underline{V}}$. One imposes the orthonormality relations

$$\underline{\underline{V}}^T \underline{\underline{T}}\ \underline{\underline{V}} = \underline{\underline{1}}$$

and verifies

$$\underline{\underline{V}}^T \underline{\underline{U}}\ \underline{\underline{V}} = \underline{\underline{\Omega}}^2$$

i.e. $\underline{\underline{U}}$ and $\underline{\underline{T}}$ are diagonalized simultaneously.

Details: [TM], Chap. 12.4, [GPS], Chap. 6

Appendix A

Supplementary material

A.1 Energy conservation of a conservative N-particle system

$$
\begin{aligned}
\triangleleft \qquad\qquad \dot{\mathbf{p}}_k &= \mathbf{F}_k + \sum_{i=1}^{N} \mathbf{f}_{ki} \\
\longrightarrow \sum_{i,k} \mathbf{f}_{ki} \cdot \mathbf{v}_k &= \sum_k m_k \cdot \dot{\mathbf{v}}_k \cdot \mathbf{v}_k - \sum_k \mathbf{F}_k \cdot \mathbf{v}_k \\
\text{rhs :} \qquad &= \frac{1}{2}\frac{d}{dt}\sum_k m_k \mathbf{v}_k^2 + \sum_k \nabla_k U_k \cdot \frac{d\mathbf{r}_k}{dt} \\
&= \frac{d}{dt}\sum_k \Big(\frac{m_k}{2}\mathbf{v}_k^2 + U_k(\mathbf{r}_k(t))\Big) \\
&= \frac{d}{dt}\sum_k \Big(T_k + U_k\Big) \\
\text{lhs :} \qquad &= \frac{1}{2}\sum_{i\neq k}\Big(\mathbf{f}_{ki}\cdot\mathbf{v}_k + \mathbf{f}_{ik}\cdot\mathbf{v}_i\Big) \\
&= \frac{1}{2}\sum_{i\neq k}\mathbf{f}_{ki}\cdot\big(\mathbf{v}_k - \mathbf{v}_i\big)
\end{aligned}
$$

$$
\begin{array}{llrcl}
\text{define :} & & \mathbf{r}_{ki} & := & \mathbf{r}_k - \mathbf{r}_i \\
& & \mathbf{v}_{ki} & := & \mathbf{v}_k - \mathbf{v}_i
\end{array}
$$

$$
\begin{array}{rcl}
\hookrightarrow \quad \nabla_k \bar{U}_{ki} & = & \nabla_{ki}\bar{U}_{ki} = -\nabla_i \bar{U}_{ik} \\
\hookrightarrow \mathbf{f}_{ki} & = & -\nabla_k \bar{U}_{ki} = -\nabla_{ki}\bar{U}_{ki}
\end{array}
$$

$$
\begin{array}{rcl}
\longrightarrow \quad \text{lhs} & = & \frac{1}{2}\sum_{i\neq k} \mathbf{f}_{ki}\cdot \mathbf{v}_{ki} \\
& = & -\frac{1}{2}\sum_{i\neq k} \nabla_{ki}\bar{U}_{ki}\cdot \frac{d\mathbf{r}_{ki}}{dt} \\
& = & -\frac{d}{dt}\frac{1}{2}\sum_{i\neq k} \bar{U}_{ki}\big(\mathbf{r}_{ki}(t)\big)
\end{array}
$$

$$
\begin{array}{rcl}
\longrightarrow \quad \text{lhs} - \text{rhs} & = & 0 \\
\frac{d}{dt}\Big(\sum_k (T_k + U_k) + \frac{1}{2}\sum_{i\neq k}\bar{U}_{ki}\Big) & = & 0 \\
\frac{d}{dt}\Big(T + U\Big) & = & 0.
\end{array}
$$

A.2 Does S assume a minimum for the actual path (i.e., is the stationary point always a minimum)?

(i) Prelude: $U(x) = 0$

$$L = \frac{m}{2}\dot{x}^2 = T$$

The (Lagrangian) equation of motion $\ddot{x} = 0$ implies $\dot{x} = v_0 = \text{const.}$ Recall the definition of neighbouring paths

$$
\begin{array}{rcl}
x(\alpha, t) & = & x(t) + \alpha\eta(t) \\
\dot{x}(\alpha, t) & = & \dot{x}(t) + \alpha\dot{\eta}(t)
\end{array}
$$

$$\text{with} \qquad \eta(t_1) = \eta(t_2) = 0$$

$$\triangleleft \qquad \begin{aligned} S(\alpha) &= \frac{m}{2}\int_{t_1}^{t_2} \dot{x}^2(\alpha,t)\ dt \\ &= \frac{m}{2}\int_{t_1}^{t_2} \dot{x}^2(t)\ dt + m\alpha\int_{t_1}^{t_2} \dot{x}(t)\dot{\eta}(t)\ dt + \frac{m}{2}\alpha^2\int_{t_1}^{t_2} \dot{\eta}^2(t)\ dt \\ &= S_{\text{actual}} + m\alpha v_0 \int_{t_1}^{t_2} \dot{\eta}(t)\ dt + \frac{m}{2}\alpha^2\int_{t_1}^{t_2} \dot{\eta}^2(t)\ dt \\ &= S_{\text{actual}} + m\alpha v_0 \eta(t)\Big|_{t_1}^{t_2} + \frac{m}{2}\alpha^2\int_{t_1}^{t_2} \dot{\eta}^2(t)\ dt \\ &= S_{\text{actual}} + \beta \end{aligned}$$

with $\beta = \frac{m}{2}\alpha^2\int_{t_1}^{t_2}\dot{\eta}^2 dt > 0$. We do have a minimum, since $S(\alpha) > S_{\text{actual}}$.

(ii) Let's add a potential

$$L = \frac{m}{2}\dot{x}^2 - U(x)$$

and Taylor-expand U around the stationary point

$$\begin{aligned} U(x(\alpha,t)) &= U(x)\big|_{\alpha=0} + \frac{dU}{dx}\Big|_{\alpha=0}\alpha\eta + \frac{1}{2}\frac{d^2U}{dx^2}\Big|_{\alpha=0}\alpha^2\eta^2 + \dots \\ &= U(x) - F(x)\alpha\eta - \frac{1}{2}F'(x)\alpha^2\eta^2 + \dots \end{aligned}$$

insert this into S:

$$\begin{aligned} S(\alpha) &= \int_{t_1}^{t_2} L(x(\alpha,t),\dot{x}(\alpha,t))\ dt \\ &= \frac{m}{2}\int_{t_1}^{t_2} \dot{x}^2(t)\ dt + m\alpha\int_{t_1}^{t_2} \dot{x}(t)\dot{\eta}(t)\ dt + \frac{m}{2}\alpha^2\int_{t_1}^{t_2} \dot{\eta}^2(t)\ dt \\ &\quad - \int_{t_1}^{t_2} U(x(t))dt + \alpha\int_{t_1}^{t_2} F(x(t))\eta(t)dt + \frac{\alpha^2}{2}\int_{t_1}^{t_2} F'(x(t))\eta^2(t)dt + O(\alpha^3) \\ &= \int_{t_1}^{t_2}\Big(\frac{m}{2}\dot{x}^2 - U(x)\Big)\ dt + \alpha\int_{t_1}^{t_2}\left(m\dot{x}\dot{\eta} + F(x)\eta\right)\ dt \\ &\quad + \frac{\alpha^2}{2}\int_{t_1}^{t_2}\left(m\dot{\eta}^2 + F'(x)\eta^2\right)\ dt + O(\alpha^3) \\ &= S_{\text{actual}} + \alpha m\int_{t_1}^{t_2}(\dot{x}\dot{\eta} + \ddot{x}\eta)\ dt + \frac{\alpha^2}{2}\int_{t_1}^{t_2}\left(m\dot{\eta}^2 + F'(x)\eta^2\right)\ dt + O(\alpha^3) \end{aligned}$$

$$\begin{aligned} I_1 &= \int_{t_1}^{t_2} (\dot{x}\dot{\eta} + \ddot{x}\eta)dt = \int_{t_1}^{t_2} \dot{x}\dot{\eta}dt + \dot{x}\eta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{x}\dot{\eta}dt = 0 \\ I_2 &= \int_{t_1}^{t_2} \left(m\dot{\eta}^2 + F'(x)\eta^2\right)\ dt = m\int_{t_1}^{t_2} \dot{\eta}^2 dt + \int_{t_1}^{t_2} F'(x)\eta^2 dt \end{aligned}$$

the term linear in α always vanishs, but the second-order term can be anything, since the second integral in I_2 can be positive or negative (or zero). Let's check two simple examples:

Example 1: homogeneous gravitational field

$$F = -mg \qquad \rightarrow F'(x) = 0$$

$$\hookrightarrow I_2 = m\int_{t_1}^{t_2} \dot{\eta}^2 dt > 0$$

in this case we have a minimum, since $S(\alpha) > S_{\text{actual}}$.

Example 2: linear force

$$F = kx \qquad \rightarrow F'(x) = k > 0$$

$$\hookrightarrow I_2 = \int_{t_1}^{t_2} (m\dot{\eta}^2 + k\eta^2)dt > 0$$

this also yields a minimum, but for the harmonic oscillator i.e., for $F = -kx$ we can't draw any conclusion from our expression for I_2. Indeed, one can show that—for long enough time intervals—the stationary point of the action is a saddle point in this case.

A.3 Differential constraints

m differential constraints can be characterized by

$$\sum_{j=1}^{3N} a_{ij}dx_j + a_{it}\ dt = 0\ ; \quad i = 1, ..., m$$

e.g. rolling disk (previous example (vii) in Chap. 2.2.1.a):

$$dx_{CM} \pm (R\cos\varphi)d\psi = 0\ , \qquad dy_{CM} \pm (R\sin\varphi)d\psi = 0$$

$$\hookrightarrow \quad a_{11} = 1\,, \qquad a_{21} = 1$$
$$a_{14} = \pm R\cos\varphi\,, \qquad a_{24} = \pm R\sin\varphi$$

all other coefficients including a_{1t} and a_{2t} are zero

$$\text{if} \qquad a_{it} \begin{cases} = 0 & \text{scleronomic} \\ \neq 0 & \text{rheonomic} \end{cases}$$

$\lessdot$ total differential of a holonomic constraint $f(x_1, ..., x_{3N}) = 0$

$$\begin{aligned} df(x_1,...,x_{3N}) &= \sum_{j=1}^{3N} \frac{\partial f}{\partial x_j}\, dx_j \\ &= \nabla f \cdot d\mathbf{r} \qquad (\text{with } \nabla = (\partial_{x_1},...,\partial_{x_{3N}})) \end{aligned}$$

$\hookrightarrow$ reobtain f by integration

$$f(x_1,...,x_{3N}) = \int df = \int \nabla f \cdot d\mathbf{r} \qquad (\text{path} - \text{independent})$$

$$\iff \quad \frac{\partial^2 f}{\partial x_j \partial x_l} = \frac{\partial^2 f}{\partial x_l \partial x_j}$$

(fulfilled, if f sufficiently well-behaved)

$$\Longrightarrow \quad \text{differential constraints} \quad \sum_{j=1}^{3N} a_{ij}\, dx_j + a_{it}\, dt \quad \text{are holonomic}$$

$$\text{if} \qquad \frac{\partial a_{ij}}{\partial x_l} = \frac{\partial a_{il}}{\partial x_j}\ ; \quad \frac{\partial a_{ij}}{\partial x_t} = \frac{\partial a_{it}}{\partial x_j}$$

proof: for holonomic constraint $f_i(x_1, ..., x_{3N}, t) = 0$

$$\begin{aligned} df_i &= \sum_{j=1}^{3N} \frac{\partial f_i}{\partial x_j}\, dx_j + \frac{\partial f_i}{\partial t}\, dt = 0 \\ &= \sum_{j=1}^{3N} a_{ij}\, dx_j + a_{it}\, dt \quad \text{with} \quad a_{ij} = \frac{\partial f_i}{\partial x_j},\ a_{it} = \frac{\partial f_i}{\partial t} \end{aligned}$$

f fulfils integrability conditions

$$\begin{array}{ccccccc} \dfrac{\partial^2 f_i}{\partial x_j \partial x_l} & = & \dfrac{\partial^2 f_i}{\partial x_l \partial x_j}\,, & \dfrac{\partial^2 f_i}{\partial x_j \partial t} & = & \dfrac{\partial^2 f_i}{\partial t \partial x_j} & \\ \| & & \| & \| & & \| & \\ \dfrac{\partial a_{il}}{\partial x_j} & = & \dfrac{\partial a_{ij}}{\partial x_l}\,, & \dfrac{\partial a_{it}}{\partial x_j} & = & \dfrac{\partial a_{ij}}{\partial t} & \text{q.e.d.} \end{array}$$

One can use the integrability conditions to check that the constraints of the rolling disk are not holonomic.

A.4 Details regaring the proof of equivalence of Newton's and Lagrange's equations of motion

Let us look more closely at the force components in

$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} = F_i + \sum_{j=1}^{3N} f_{ij}$$

Basically this is a (tedious) counting exercise:

(i) external forces:

$$\begin{aligned} \sum_{k=1}^{N} U_k(\mathbf{r}_k) &= U_1(x_1 x_2 x_3) + U_2(x_4 x_5 x_6) \\ &\qquad + \ldots + U_N(x_{3N-2} x_{3N-1} x_{3N}) \\ &\equiv U_{\text{ext}}(x_1, \ldots, x_{3N}) \end{aligned}$$

$\rightarrow$ external force on k-th mass point: $\mathbf{F}_k = -\nabla_k U_k \hookrightarrow$ m-th component of the force on k-th mass point:

$$F_k^m = -\frac{\partial}{\partial x_k^m} U_k = -\frac{\partial U_{\text{ext}}}{\partial x_k^m}\,, \quad (k = 1, ..., N;\ m = 1, 2, 3)$$

explicitly:

$$\begin{pmatrix} F_k^1 \equiv F_k^x = -\frac{\partial}{\partial x_k} U_{\text{ext}} \\ \\ F_k^2 \equiv F_k^y = -\frac{\partial}{\partial y_k} U_{\text{ext}} \\ \\ F_k^3 \equiv F_k^z = -\frac{\partial}{\partial z_k} U_{\text{ext}} \\ \end{pmatrix} \Longleftrightarrow F_i = -\frac{\partial U_{\text{ext}}}{\partial x_i} \ , \quad i = 1, ..., 3N$$

$$\begin{array}{ccccccccccc} F_1^x & F_1^y & F_1^z & F_2^x & F_2^y & F_2^z & ... & F_N^x & F_N^y & F_N^z \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & & F_{3N-2} & F_{3N-1} & F_{3N} \end{array}$$

(ii) internal forces (note that $\bar{U}_{ik} = \bar{U}_{ki}$):

$$\begin{aligned} \mathbf{f}_{ij} \ = \ \nabla_i \bar{U}_{ij} \ &= \ -\nabla_j \bar{U}_{ij} \\ \sum_{j<i}^{N} \bar{U}_{ij}(\mathbf{r}_i - \mathbf{r}_j) \ &= \ \bar{U}_{21}(x_1 x_2 x_3, x_4 x_5 x_6) \\ & \quad + \bar{U}_{31}(x_1 x_2 x_3, x_7 x_8 x_9) + ... + \bar{U}_{32}(x_4 x_5 x_6, x_7 x_8 x_9) + ... \\ & \quad + ... + \ \bar{U}_{N,N-1}(x_{3N-5} x_{3N-4} x_{3N-3}, x_{3N-2} x_{3N-1} x_{3N}) \end{aligned}$$

some examples:

$$\begin{aligned} -\frac{\partial \bar{U}}{\partial x_1} \ &= \ -\frac{\partial}{\partial x_1}(\bar{U}_{21} + \bar{U}_{31} + ... + \bar{U}_{N1}) \\ &= \ -\frac{\partial}{\partial x_1}(\bar{U}_{12} + \bar{U}_{13} + ... + \bar{U}_{1N}) \ = \ f_1^x \ = \ \sum_{j=1}^{N} f_{1j}^x \\ &= \ \sum_{j=1}^{3N} f_{1j} \\ -\frac{\partial \bar{U}}{\partial x_5} \ &= \ -\frac{\partial}{\partial x_5}(\bar{U}_{21} + \bar{U}_{23} + \bar{U}_{24} + ... + \bar{U}_{2N}) \\ &= \ f_2^y \ = \ \sum_{j=1}^{N} f_{2j}^y \ = \ \sum_{j=1}^{3N} f_{5j} \\ &\ \ \vdots \end{aligned}$$

$$
\begin{aligned}
-\frac{\partial \bar{U}}{\partial x_{3N}} &= -\frac{\partial}{\partial x_{3N}}(\bar{U}_{N1} + \bar{U}_{N2} + ... + \bar{U}_{N,N-1}) \\
&= f_N^z = \sum_{j=1}^{N} f_{Nj}^z = \sum_{j=1}^{3N} f_{3Nj}
\end{aligned}
$$

$$
\hookrightarrow \quad \underline{\text{in general :}} \quad -\frac{\partial \bar{U}}{\partial x_i} = ... = \sum_{j=1}^{3N} f_{ij}
$$

with matrix $f_{ji}(3N \times 3N)$ of the following structure:

$i \setminus j$	1	2	3	4	5	6	7	..	..	3N-2	3N-1	3N
1	0	0	0	x	0	0	x	..	..	x	0	0
2	0	0	0	0	x	0	0	..	..	0	x	0
3	0	0	0	0	0	x	0	..	..	0	0	x
4	x	0	0	0	0	0	x	..	..	x	0	0
5	0	x	0	0	0	0	0	..	..	0	x	0
6	0	0	x	0	0	0	0	..	..	0	0	x
7	x	0	0	x	0	0	0	0	0	.		
⋮	⋮			⋮			0	..	..	⋮		
3N												

summary: $f_{ij} \neq 0$ if $1 \leq i = j \pm 3n \leq 3N$; $(n = 1, ..., N-1)$

$$
\begin{aligned}
\text{it follows :} \qquad & -\frac{\partial \bar{U}}{\partial x_i} = \sum_{j=1}^{3N} f_{ij} \ , \qquad i = 1, ..., 3N \\
\Longleftrightarrow \quad & -\nabla_i \bar{U}_{ij} = \mathbf{f}_{ij} \ , \qquad i, j = 1, ..., N \\
\Longrightarrow \quad & \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \dot{p}_i - F_i - \sum_{j=1}^{3N} f_{ij} = 0 \\
\Longleftrightarrow \quad & \dot{\mathbf{p}}_k = \mathbf{F}_k + \sum_{j=1}^{N} \mathbf{f}_{kj} \ , \qquad k = 1, ..., N
\end{aligned}
$$

A.5 Some details regarding rigid body dynamics

Any rotation about a fixed point can be represented by a series of three rotations about designated axes through the Eulerian angles (α, β, γ).

$$\begin{array}{ccccccc} \text{coordinates} & & & & & & \text{coordinates} \\ \text{wrt. } S_f & & & & & & \text{wrt. } S_b \\ (x_1', x_2', x_3') & \xrightarrow{\alpha} & (x_1'', x_2'', x_3'') & \xrightarrow{\beta} & (x_1''', x_2''', x_3''') & \xrightarrow{\gamma} & (x_1, x_2, x_3) \end{array}$$

$$\mathbf{r}_{S_b} = \underline{\underline{D}}_{\,\gamma} \, \underline{\underline{D}}_{\,\beta} \, \underline{\underline{D}}_{\,\alpha} \, \mathbf{r}_{S_f} = \underline{\underline{D}} \, \mathbf{r}_{S_f}$$

Each rotation is characterized by a rotation matrix. The product of these matrices (note their order) characterizes the resulting rotation.

<u>rotation 1:</u> rotate reference system S_f counterclockwise through α about x_3'-axis:

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \qquad (\mathbf{r}'' = \underline{\underline{D}}_{\,\alpha} \, \mathbf{r}_{S_f})$$

<u>rotation 2:</u> rotate the new system (S'') counterclockwise through β about x_1''-axis:

$$\begin{pmatrix} x_1''' \\ x_2''' \\ x_3''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} \qquad (\mathbf{r}''' = \underline{\underline{D}}_{\,\beta} \, \mathbf{r}'')$$

<u>rotation 3:</u> rotate the new system (S''') counterclockwise through γ about x_3'''-axis:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1''' \\ x_2''' \\ x_3''' \end{pmatrix} \qquad (\mathbf{r}_{S_b} = \underline{\underline{D}}_{\,\gamma} \, \mathbf{r}''')$$

multiply the matrices:

$$\underline{\underline{D}} = \underline{\underline{D}}_{\gamma}\,\underline{\underline{D}}_{\beta}\,\underline{\underline{D}}_{\alpha}$$
$$= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\ -\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{pmatrix}$$

Since the total rotation consists of three rotations, the angular velocity vector $\boldsymbol{\omega}$ (which describes the orientation of the body system) is the sum of the three corresponding angular velocity vectors:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\gamma$$

We need to find the components of these vectors in the body system.

(i) $\boldsymbol{\omega}_\alpha$ points in x_3'-direction in the fixed system (cf. rotation 1). The components of this vector in the body system are obtained from applying $\underline{\underline{D}} = \underline{\underline{D}}_{\gamma}\,\underline{\underline{D}}_{\beta}\,\underline{\underline{D}}_{\alpha}$:

$$\hookrightarrow \quad \boldsymbol{\omega}_\alpha\Big|_{S_b} = \begin{pmatrix} \dot\alpha_1 \\ \dot\alpha_2 \\ \dot\alpha_3 \end{pmatrix} = \underbrace{\underline{\underline{D}}\begin{pmatrix} 0 \\ 0 \\ \dot\alpha \end{pmatrix}}_{\| \atop \underline{\underline{D}}_{\gamma}\,\underline{\underline{D}}_{\beta}\begin{pmatrix} 0 \\ 0 \\ \dot\alpha \end{pmatrix}} = \begin{pmatrix} \dot\alpha\sin\beta\sin\gamma \\ \dot\alpha\sin\beta\cos\gamma \\ \dot\alpha\cos\beta \end{pmatrix}$$

(note that the componentes of $\boldsymbol{\omega}_\alpha$ do not change during the first rotation)

(ii) $\boldsymbol{\omega}_\beta$ points in $x_1'' = x_1'''$ direction in the systems S'' and S'''. The components with respect to the body system are obtained from applying $\underline{\underline{D}}_{\gamma}$:

$$\boldsymbol{\omega}_\beta\Big|_{S_b} = \begin{pmatrix} \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{pmatrix} = \underline{\underline{D}}_{\gamma}\begin{pmatrix} \dot\beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot\beta\cos\gamma \\ -\dot\beta\sin\gamma \\ 0 \end{pmatrix}$$

(iii) $\boldsymbol{\omega}_\gamma$ points in $x_3''' = x_3$ direction in the S''' and the body system:

$$\boldsymbol{\omega}_\gamma\Big|_{S_b} = \begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \underline{\underline{D}}_\gamma \begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix}$$

Collecting terms yields $\omega_k = \dot{\alpha}_k + \dot{\beta}_k + \dot{\gamma}_k$

$$\begin{aligned} \omega_1 &= \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma \\ \omega_2 &= \dot{\alpha}\sin\beta\cos\gamma - \dot{\beta}\sin\gamma \\ \omega_3 &= \dot{\alpha}\cos\beta + \dot{\gamma} \end{aligned}$$

In the principal axes body system we have

$$\begin{aligned} T_{rot} &= \frac{1}{2}\sum_k I_k\omega_k^2 \\ &= \frac{1}{2}\Big(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\Big) \\ &= \frac{1}{2}\Big\{I_1\big(\dot{\alpha}^2\sin^2\beta\sin^2\gamma + \dot{\beta}^2\cos^2\gamma + 2\dot{\alpha}\dot{\beta}\sin\beta\sin\gamma\cos\gamma\big) \\ &\quad + I_2\big(\dot{\alpha}^2\sin^2\beta\cos^2\gamma + \dot{\beta}^2\sin^2\gamma - 2\dot{\alpha}\dot{\beta}\sin\beta\sin\gamma\cos\gamma\big) \\ &\quad + I_3\big(\dot{\alpha}^2\cos^2\beta + \dot{\gamma}^2 + 2\dot{\alpha}\dot{\gamma}\cos\beta\big)\Big\} \\ &= T_{rot}\Big(\beta\gamma, \dot{\alpha}\dot{\beta}\dot{\gamma}\Big) \end{aligned}$$

We are after the Lagrangian equations of motion for

$$L_{rot} = T_{rot} - U = T_{rot}\Big(\beta\gamma, \dot{\alpha}\dot{\beta}\dot{\gamma}\Big) - U\Big(\alpha\beta\gamma\Big)$$

$$\frac{d}{dt}\left(\frac{\partial T_{rot}}{\partial \dot{q}_k}\right) - \frac{\partial T_{rot}}{\partial q_k} = -\frac{\partial U}{\partial q_k} \qquad \begin{pmatrix} q_1 = \alpha \\ q_2 = \beta \\ q_3 = \gamma \end{pmatrix}$$

Let's look at one coordinate explicitly:

$\underline{q_3 = \gamma}$:

- $\dfrac{\partial T_{rot}}{\partial \dot{\gamma}} = I_3\big(\dot{\gamma} + \dot{\alpha}\cos\beta\big) \qquad \hookrightarrow \qquad \dfrac{d}{dt}\dfrac{\partial T_{rot}}{\partial \dot{\gamma}} = I_3\Big(\ddot{\alpha}\cos\beta - \dot{\alpha}\dot{\beta}\sin\beta + \ddot{\gamma}\Big)$

- $$\frac{\partial T_{rot}}{\partial \gamma} = I_1\Big(\dot\alpha^2\sin^2\beta\sin\gamma\cos\gamma - \dot\beta^2\sin\gamma\cos\gamma + \dot\alpha\dot\beta\sin\beta\cos^2\gamma - \dot\alpha\dot\beta\sin\beta\sin^2\gamma\Big)$$
$$+ \quad I_2\Big(- \dot\alpha^2\sin^2\beta\sin\gamma\cos\gamma + \dot\beta^2\sin\gamma\cos\gamma - \dot\alpha\dot\beta\sin\beta\cos^2\gamma + \dot\alpha\dot\beta\sin\beta\sin^2\gamma\Big)$$

$\hookrightarrow$ EoM:
$$\boxed{\begin{aligned} &I_3\big(\ddot\alpha\cos\beta - \dot\alpha\dot\beta\sin\beta + \ddot\gamma\big) \\ &- \big(I_1 - I_2\big)\Big\{\big(\dot\alpha^2\sin^2\beta - \dot\beta^2\big)\sin\gamma\cos\gamma \\ &\quad + \dot\alpha\dot\beta\sin\beta\big(\cos^2\gamma - \sin^2\gamma\big)\Big\} \;=\; -\frac{\partial U}{\partial\gamma}\end{aligned}}$$

One can show that this equation can be written in the form

$$\boxed{I_3\dot\omega_3 - \big(I_1 - I_2\big)\omega_1\omega_2 = -\frac{\partial U}{\partial \gamma}}$$

let's check:

$$\begin{aligned} \omega_1\omega_2 &= \big(\dot\alpha\sin\beta\sin\gamma + \dot\beta\cos\gamma\big)\big(\dot\alpha\sin\beta\cos\gamma - \dot\beta\sin\gamma\big) \\ &= \dot\alpha^2\sin^2\beta\sin\gamma\cos\gamma + \dot\alpha\dot\beta\sin\beta\cos^2\gamma \\ &\quad - \dot\beta^2\sin\gamma\cos\gamma - \dot\alpha\dot\beta\sin\beta\sin^2\gamma \\ &= \big(\dot\alpha^2\sin\beta - \dot\beta^2\big)\sin\gamma\cos\gamma + \dot\alpha\dot\beta\sin\beta\big(\cos^2\gamma - \sin^2\gamma\big)\end{aligned}$$

Similar (quite lengthy) calculations yield EoMs for $q_1 = \alpha$ and $q_2 = \beta$:

$\underline{q_1 = \alpha}$:

$$\begin{aligned} &\ddot\alpha\Big\{\big(I_1\sin^2\gamma + I_2\cos^2\gamma\big)\sin^2\beta + I_3\cos^2\beta\Big\} \\ &+ 2\dot\alpha\dot\beta\big(I_1\sin^2\gamma + I_2\cos^2\gamma - I_3\big)\sin\beta\cos\beta \\ &+ 2\dot\alpha\dot\beta\big(I_1 - I_2\big)\sin^2\beta\sin\gamma\cos\gamma + \ddot\beta\big(I_1 - I_2\big)\sin\beta\sin\gamma\cos\gamma \\ &+ \dot\beta^2\big(I_1 - I_2\big)\cos\beta\sin\gamma\cos\gamma \\ &+ \dot\beta\dot\gamma\Big\{\big(I_1 - I_2\big)\big(\cos^2\gamma - \sin^2\gamma\big) - I_3\Big\}\sin\beta + \ddot\gamma I_3\cos\beta \;=\; -\frac{\partial U}{\partial\alpha}\end{aligned}$$

$\underline{q_2 = \beta}$:

$$
\begin{aligned}
\ddot{\alpha}\big(I_1 - I_2\big)\sin\beta\sin\gamma\cos\gamma & \\
+\ \dot{\alpha}^2\big(I_3 - I_1\sin^2\gamma - I_2\cos^2\gamma\big)\sin\beta\cos\beta & \\
+\ \dot{\alpha}\dot{\gamma}\Big\{I_3 + \big(I_1 - I_2\big)\big(\cos^2\gamma - \sin^2\gamma\big)\Big\}\sin\beta & \\
+\ddot{\beta}\big(I_1\cos^2\gamma + I_2\sin^2\gamma\big) - 2\dot{\beta}\dot{\gamma}\big(I_1 - I_2\big)\sin\gamma\cos\gamma &= -\frac{\partial U}{\partial\beta}
\end{aligned}
$$

Let's consider linear combinations of the three EoMs:

$$
\begin{aligned}
I_1\dot{\omega}_1 - \big(I_2 - I_3\big)\omega_2\omega_3 &= \frac{1}{\sin\beta}\Big(-\sin\gamma\frac{\partial U}{\partial\alpha} - \sin\beta\cos\gamma\frac{\partial U}{\partial\beta} + \cos\beta\sin\gamma\frac{\partial U}{\partial\gamma}\Big) \\
&\equiv N_1
\end{aligned}
$$

$$
\begin{aligned}
I_2\dot{\omega}_2 - \big(I_3 - I_1\big)\omega_3\omega_1 &= \frac{1}{\sin\beta}\Big(-\cos\gamma\frac{\partial U}{\partial\alpha} + \sin\beta\sin\gamma\frac{\partial U}{\partial\beta} + \cos\beta\cos\gamma\frac{\partial U}{\partial\gamma}\Big) \\
&\equiv N_2
\end{aligned}
$$

<u>Summary</u> (using $N_3 = -\frac{\partial U}{\partial\gamma}$):

$$
\left.
\begin{aligned}
I_1\dot{\omega}_1 - \big(I_2 - I_3\big)\omega_2\omega_3 = N_1 \\
I_2\dot{\omega}_2 - \big(I_3 - I_1\big)\omega_3\omega_1 = N_2 \\
I_3\dot{\omega}_3 - \big(I_1 - I_2\big)\omega_1\omega_2 = N_3
\end{aligned}
\right\}\quad \text{Euler equations}
$$

$$
\text{i.e.}\quad \sum_k \varepsilon_{ijk}\big(I_k\dot{\omega}_k - N_k\big) - \big(I_i - I_j\big)\omega_i\omega_j = 0
$$

$$
\text{with}\qquad \epsilon_{ijk} = \begin{cases} 1 & (ijk)\ \text{even permutation of}\ (1,2,3) \\ -1 & (ijk)\ \text{odd permutation of}\ (1,2,3) \\ 0 & \text{else} \end{cases}
$$

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